10.3: The Integral and Comparison Tests; Estimating Sums

QUESTION: For what values of \( p \) the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent?

If \( p = 1 \) then \( \sum_{n=1}^{\infty} \frac{1}{n} \) is called harmonic series.

THE INTEGRAL TEST Suppose \( f \) is a continuous, positive, decreasing function on \([1, \infty)\) and let \( a_n = f(n) \). Then the series \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if the improper integral \( \int_1^{\infty} f(x) \, dx \) is convergent. In other words:

**FACT:** The \( p \)-series, \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), converges if \( p > 1 \) and diverges if \( p \leq 1 \).

EXAMPLE 1. Determine if the following series is convergent or divergent:

(a) \( \sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}} \)
(b) \[\sum_{n=2}^{\infty} \frac{1}{n \ln n}\]

A disadvantage of the Integral Test: it does force us to do improper integrals which are in some cases may be impossible to determine the convergence of. For example, consider

[\sum_{n=0}^{\infty} \frac{1}{4^n + n^4}]
THE COMPARISON TEST Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n \leq b_n$ for all $n$.

- If $\sum b_n$ is convergent then $\sum a_n$ is also convergent.
- If $\sum a_n$ is divergent then $\sum b_n$ is also divergent.

EXAMPLE 2. Determine if the following series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{n^4 + 4}{n^6 + 6}$

(b) $\sum_{n=2}^{\infty} \frac{n^{2000}}{n^{2001} - \sin^{2000} n}$

(c) $\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2 \sqrt{n}}$
(d) \[ \sum_{n=1}^{\infty} \frac{5^n + 1}{4^n} \]

(e) \[ \sum_{n=0}^{\infty} \frac{1}{4^n + n^4} \]

**WARNING:** Distinguish between \[ \sum_{n=1}^{\infty} n^p \] and \[ \sum_{n=1}^{\infty} p^n \].

In some cases inequalities are useless. For example, for the series
\[ \sum_{n=0}^{\infty} \frac{1}{4^n - n} \]
we have
**THE LIMIT COMPARISON TEST** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]
where $c$ is a finite number and $c > 0$, then either both series converge or both diverge.

**EXAMPLE 3.** Determine if the following series is convergent or divergent:

(a) $\sum_{n=1}^{\infty} \frac{1}{4^n - n}$

(b) $\sum_{n=2}^{\infty} \frac{n^2 + n}{\sqrt{n^5 + n^3}}$
Illustration: Why $c$ in the Limit Comparison Test must be positive and finite:

REMAINDER ESTIMATE FOR THE INTEGRAL TEST

If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_n^{\infty} f(x) \, dx$$

which implies

$$s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx$$
EXAMPLE 4. Given $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(a) Approximate the sum of the series by using the sum of the first 10 terms.

(b) Estimate the error.

(c) How many terms are required to ensure that the sum is accurate to within 0.0005?
EXAMPLE 5. Given
\[ \sum_{n=1}^{\infty} \frac{1 + \sin n}{2n^3}. \]

(a) Prove the convergence.

(b) By comparison the series to a p-series, estimate the error in using \( s_{100} \) to approximate the sum of the series,