7.2: VOLUME

A simple type of solid: right cylinder

Let $S$ be any solid. The intersection of $S$ with a plane is a plane region that is called a **cross-section** of $S$.

$P_x$ is a plane perpendicular to $x$-axis and passing through $x$.

$A(x)$ is the area of cross-section obtained as intersection of $S$ and $P_x$, $a \leq x \leq b$.

(Think of slicing a loaf of bread.)
DEFINITION 1. Let $S$ be a solid that lies between the planes $P_a$ and $P_b$. Then the volume of $S$ is

$$V = \lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x^*_i) \Delta x_i =$$

Important to remember: $A(x)$ is the area of a moving cross-section obtained by slicing through $x$ perpendicular to the $x$-axis.

EXAMPLE 2. Find the volume of the cap of a ball with radius 3 and height 1.
Volumes of Solids of Revolution (Disk Method)

Consider the plane region $D$ bounded by the curves $y = f(x), y = 0, x = a, x = b$, i.e.

$$D =$$

Rotate $D$ about a given axis to get the solid of revolution $S$:

**PROBLEM**: Determine the volume of solid of revolution.

**Solution**: Using cross-sectional areas (disk method)
EXAMPLE 3. Determine the volume of the solid obtained by rotating the region 

\[ D = \{(x, y) : 1 \leq x \leq 4, 0 \leq y \leq x^2 - 4x + 5\} \]

about the x-axis.

EXAMPLE 4. Determine the volume of the solid obtained by rotating the region enclosed by 

\[ y = x^3, \quad y = 8, \quad x = 0 \]

about the y-axis.
EXAMPLE 5. Determine the volume of the solid obtained by rotating the region enclosed by \( y = \ln x, \ y = 0, \ y = 5 \ x = 0 \) about the \( y \)-axis.

EXAMPLE 6. Determine the volume of the solid obtained by rotating the region enclosed by the curves \( y = \sqrt[3]{x}, \ x = 8, \ y = 0 \) about the line \( x = 8 \).
EXAMPLE 7. Determine the volume of the solid obtained by rotating the region enclosed by \( y = \tan x \), \( y = 1 \) and the \( y \)-axis about the line \( y = 1 \).

\[
\begin{array}{c|c}
\text{SUMMARY (Disk Method)} \\
\hline
\text{Rotation about a horizontal axis (} y = k \text{): } V = \int_a^b A(x) \, dx \\
\text{Rotation about a vertical axis (} x = k \text{): } V = \int_a^b A(y) \, dy \\
\text{Cross sections are orthogonal to the axis of rotating.}
\end{array}
\]
**Washer Method**

Use it when the cross-sections orthogonal to the axis of rotating of a solid of revolution are in the shape of a washer (ring).

The area of a ring:

\[
\pi (R^2 - r^2)
\]

**EXAMPLE 8.** Let \( D \) be the plane region that lies in the first quadrant and enclosed by \( y = \sqrt{x} \) and \( y = \frac{x}{4} \).

(a) Determine the volume of the solid obtained by rotating the region \( D \) about the y-axis.
(b) Determine the volume of the solid obtained by rotating the region $D$ about the $x$-axis.

EXAMPLE 9. Let $D$ be the region enclosed by $y = x$ and $y = x^2$.

(a) Determine the volume of the solid obtained by rotating the region $D$ about the line $x = -1$. 
(b) Determine the volume of the solid obtained by rotating the region $D$ about the line $y = 2$.

More general case: Cross Sections other than Circles

Use the basic formula:

$$ V = \int_a^b A(x) \, dx $$

EXAMPLE 10. Find the volume of the solid whose base is a disk with radius 5 and the cross sections perpendicular to the $y$-axis are equilateral triangles.
EXAMPLE 11. The base of the solid $S$ is the triangular region with the vertices $(0,0), (1,0)$ and $(0,1)$. Find the volume of $S$ if the cross sections perpendicular to the $x$-axis are semicircles with diameters on the base.