18: Definition of the Laplace Transform (section 6.1)

1. Remind the Improper Integral (type I):

\[
\int_0^\infty \phi(t)dt = \lim_{A \to \infty} \int_0^A \phi(t)dt
\]

2. **DEFINITION of LAPLACE TRANSFORM** Let \( f(t) \) be a function defined for \( t \geq 0 \). Then the integral

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt
\]

is said to be the **Laplace Transform** of \( f \), provided that the integral converges. Note that when the integral (1) converges the result is a function of \( s \).

Below we use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace Transform:

\[
\mathcal{L}\{f(t)\} = F(s),\quad \mathcal{L}\{g(t)\} = G(s),\quad \mathcal{L}\{y(t)\} = Y(s), \text{etc.}
\]

3. Example: Apply the above definition to evaluate Laplace Transform of the following functions:

   (a) \( f(t) = 1 \)

   (b) \( f(t) = e^{5t} \)

4. \( \mathcal{L} \) is a Linear Transform:

\[
\mathcal{L}\{\alpha f + \beta g\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}.
\]

5. How Laplace Transform might be useful in solving DE? Key property: Under some natural conditions on a function \( f \) we have transform of a derivative

\[
\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).
\]

**Illustration:** We already know that \( y(t) = 10e^{-5t} \) is solution of the IVP:

\[
y' + 5y = 0,\quad y(0) = 10.
\]

Now solve it using Laplace Transform.

6. Example: Evaluate Laplace Transform of the following functions:

   (a) \( f(t) = \sin(4t) \)

   (b) \( f(t) = \cos(at) \)
7. Transforms of some basic functions
\[ L\{1\} = \frac{1}{s} \]
\[ L\{e^{at}\} = \frac{1}{s-a} \]
\[ L\{\sin at\} = \frac{a}{s^2 + a^2} \]
\[ L\{\cos at\} = \frac{s}{s^2 + a^2} \]
\[ L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots \]

8. Translation in s property:
\[ L\{e^{\alpha t} f(t)\} = F(s - \alpha) \]

9. EXAMPLE Evaluate
   (a) \[ L\{e^{\alpha t} \sin \beta t\} \]
   (b) \[ L\{e^{\alpha t} \cos \beta t\} \]

10. Laplace transform of the derivative: Under some natural conditions on a function f
\[ L\{f'(t)\} = sL\{f(t)\} - f(0) \]
   More generally,
\[ L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0) \]
\[ L\{f^{(n)}(t)\} = s^nL\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0) \]

11. EXAMPLE Solve for \(Y(s)\), the Laplace transform of the solution \(y(t)\) to the given initial value problem:
\[ y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12. \]

12. Derivative of Laplace transform:
\[ L\{t^n f(t)\} = (-1)^n F^{(n)}(s). \]

13. EXAMPLE Evaluate \(L\{t^n e^{\alpha t}\}\)

19: Solution of Initial Value Problems (sec. 6.2)

1. INVERSE LAPLACE TRANSFORMS: If \(F(s)\) represents the Laplace Transform of \(f(t)\), i.e.
\[ L\{f(t)\} = F(s), \]
then we say that \(f(t)\) is the **inverse Laplace Transform** of \(F(s)\) and write
\[ f(t) = L^{-1}\{F(s)\}. \]

---

\(^1\)s is sufficiently restricted to guarantee the convergence of the appropriate Laplace Transform.
2. Some Inverse Transforms:

Transform

\[ \mathcal{L} \{ 1 \} = \frac{1}{s} \]

\[ \mathcal{L} \{ t^n \} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots \]

\[ \mathcal{L} \{ e^{at} \} = \frac{1}{s - a} \]

\[ \mathcal{L} \{ \sin at \} = \frac{a}{s^2 + a^2} \]

\[ \mathcal{L} \{ \cos at \} = \frac{s}{s^2 + a^2} \]

\[ \mathcal{L} \{ t^n e^{at} \} = \frac{n!}{(s - \alpha)^{n+1}} \]

Inverse Transform

\[ \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = 1 \]

\[ \mathcal{L}^{-1} \left\{ \frac{1}{s^n} \right\} = \frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots \]

\[ \mathcal{L}^{-1} \left\{ \frac{1}{s - a} \right\} = e^{at} \]

\[ \mathcal{L}^{-1} \left\{ \frac{a}{s^2 + a^2} \right\} = \sin at \]

\[ \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} = \cos at \]

\[ \mathcal{L}^{-1} \left\{ \frac{1}{(s - \alpha)^n} \right\} = \frac{t^{n-1}e^{at}}{(n-1)!} \]

See Table on the page 317 in the Textbook (or Appendix 2) for more cases.

3. \( \mathcal{L}^{-1} \) is a Linear Transform:

\[ \mathcal{L}^{-1} \{ \alpha f + \beta g \} = \alpha \mathcal{L}^{-1} \{ f \} + \beta \mathcal{L}^{-1} \{ g \} . \]

4. Note that it often happens that a function of \( s \) under consideration does not match exactly the form of a Laplace Transform \( F(s) \) in the table. In this cases you need to “fix up” the function of \( s \). Helpful strategies:

- multiply/divide by an appropriate constant
- use termwise division
- use Partial Fractions (See Appendix 1: Inverse Laplace transform of rational functions using Partial Fraction Decomposition)

5. Example. Evaluate

(a) \( \mathcal{L}^{-1} \left\{ \frac{2s + 3}{s^2 + 5s + 6} \right\} \)

(b) \( \mathcal{L}^{-1} \left\{ \frac{2s^2 - 3s + 5}{(s - 3)^2(s + 4)} \right\} \)

(c) \( \mathcal{L}^{-1} \left\{ \frac{3s + 5}{s^2 + 6s + 34} \right\} \)
6. Consider the $n$-th order ODE

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(t) \]

subject to

\[ y(0) = \alpha_0, \quad y'(0) = \alpha_1, \quad \ldots \quad y^{(n-1)}(0) = \alpha_{n-1}. \]

Note that in the case $n = 2$ know how to solve this IVP using the Method Variation of Parameters and the Method of Undetermined Coefficients (for $g(t) = P_n(t)e^{\alpha t} \cos bt$ or $g(t) = P_n(t)e^{\alpha t} \sin bt$).

7. How to solve the given IVP using Laplace Transform:

**Step 1.** Apply Laplace Transform to both sides of the given ODE. Use linearity and other Laplace Transform properties together with the initial conditions to obtain an algebraic equation in the $s$-domain for $Y(s) = \mathcal{L}\{y(t)\}$ instead of the given ODE in the $t$-domain.

**Step 2.** Solve for $Y(s)$ the algebraic equation obtained in Step 1.

**Step 3.** Find the inverse Laplace Transform of $Y(s)$ to get $y(t)$.

8. **EXAMPLE** Solve IVP

\[ y'' - 2y' + 5y = -8e^{-t}, \quad y(0) = 2, \quad y'(0) = 12. \]

Note that we already found that

\[ Y(s) = \frac{2s^2 + 10s}{(s + 1)(s^2 - 2s + 5)}. \]

9. **EXAMPLE** Consider the IVP

\[ y'' + 4y' - 5y = te^t, \quad y(0) = 1, \quad y'(0) = 0. \] (2)

**SOLUTION** (Main Steps): Application of Laplace Transform yields:

\[ Y(s) = \frac{s^3 + 2s^2 - 7s + 5}{(s - 1)^2(s^2 + 4s - 5)} = \frac{s^3 + 2s^2 - 7s + 5}{(s - 1)^3(s + 5)} \] (3)

Partial Fraction Decomposition:

\[ \frac{s^3 + 2s^2 - 7s + 5}{(s - 1)^2(s^2 + 4s - 5)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{C}{(s - 1)^3} + \frac{D}{s + 5}, \] (4)

where

\[ A = \frac{181}{216}, \quad B = -\frac{1}{36}, \quad C = \frac{1}{6}, \quad D = \frac{35}{216}. \]

Find the inverse Laplace Transform of $Y(S)$ (use Table (see Appendix 2)):

\[ y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{181}{216}e^t - \frac{1}{36}te^t + \frac{1}{12}t^2e^t + \frac{35}{216}e^{-5t}. \] (5)

**QUESTION:** What is the general form of the solution of DE (2) by Method of Undetermined Coefficients?

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2These methods can be straightforward generalized for any $n$. 

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Appendix 1.

Inverse Laplace transform of rational functions using Partial Fraction Decomposition

Using the Laplace transform for solving linear non-homogeneous differential equation with constant coefficients and the right-hand side \( g(t) \) of the form \( h(t)e^{\alpha t} \cos \beta t \) or \( h(t)e^{\alpha t} \sin \beta t \), where \( h(t) \) is a polynomial, one needs on certain step to find the inverse Laplace transform of rational functions \( \frac{P(s)}{Q(s)} \) where \( P(s) \) and \( Q(s) \) are polynomials with \( \deg P(s) < \deg Q(s) \).

The latter can be done by means of the partial fraction decomposition that you studied in Calculus 2: One factors the denominator \( Q(s) \) as much as possible, i.e. into linear (may be repeated) and quadratic (may be repeated) factors: each linear factor correspond to a real root of \( Q(s) \) and each quadratic factor corresponding to a pair of complex conjugate roots of \( Q(s) \).

Each factor in the decomposition of \( Q(s) \) gives a contribution of certain type to the partial fraction decomposition of \( \frac{P(s)}{Q(s)} \). Below we list these contributions depending on the type of the factor and identify the inverse Laplace transform of these contributions:

Case 1 A non-repeated linear factor \((s - a)\) of \( Q(s) \) (corresponding to the root \( a \) of \( Q(s) \) of multiplicity 1) gives a contribution of the form \( \frac{A}{s - a} \). Then \( \mathcal{L}^{-1} \left\{ \frac{A}{s - a} \right\} = Ae^{at} \);

Case 2 A repeated linear factor \((s - a)^m\) of \( Q(s) \) (corresponding to the root \( a \) of \( Q(s) \) of multiplicity \( m \)) gives a contribution which is a sum of terms of the form \( \frac{A_i}{(s - a)^i} \), \( 1 \leq i \leq m \).

Then \( \mathcal{L}^{-1} \left\{ \frac{A_i}{(s - a)^i} \right\} = \frac{A_i}{(i - 1)!} t^{i-1} e^{at} \);
Case 3 A non-repeated quadratic factor \((s - \alpha)^2 + \beta^2\) of \(Q(s)\) (corresponding to the pair of complex conjugate roots \(\alpha \pm i\beta\) of multiplicity 1) gives a contribution of the form \[\frac{Cs + D}{(s - \alpha)^2 + \beta^2}.\] It is more convenient here to represent it in the following way: \[\frac{Cs + D}{(s - \alpha)^2 + \beta^2} = \frac{A(s - \alpha) + B\beta}{(s - \alpha)^2 + \beta^2}.\] Then \[L^{-1}\left\{\frac{A(s - \alpha) + B\beta}{(s - \alpha)^2 + \beta^2}\right\} = Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t;\]

Case 4 A repeated quadratic factor \(((s - \alpha)^2 + \beta^2)^m\) of \(Q(s)\) (corresponding to the pair of complex conjugate roots \(\alpha \pm i\beta\) of multiplicity \(m\)) gives a contribution which is a sum of terms of the form \[\frac{C_is + D_i}{((s - \alpha)^2 + \beta^2)^i} = \frac{A_i(s - \alpha) + B_i\beta}{((s - \alpha)^2 + \beta^2)^i},\] where \(1 \leq i \leq m.\)

The calculation of the inverse Laplace transform in this case is more involved. It can be done as a combination of the property of the derivative of Laplace transform and the notion of convolution that will be discussed in section 6.6.
### Appendix 2.

#### Table 6.2.1 Elementary Laplace Transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n$, $n = \text{positive integer}$</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$t^p$, $p &gt; -1$</td>
<td>$\frac{\Gamma(p+1)}{s^{p+1}}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$, $s &gt;</td>
</tr>
<tr>
<td>$\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$, $s &gt;</td>
</tr>
<tr>
<td>$e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$, $s &gt; a$</td>
</tr>
<tr>
<td>$e^{at} \cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n e^{at}$, $n = \text{positive integer}$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$, $s &gt; a$</td>
</tr>
<tr>
<td>$u_c(t)$</td>
<td>$\frac{e^{-cs}}{s}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$u_c(t)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
</tr>
<tr>
<td>$e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
</tr>
<tr>
<td>$f(ct)$</td>
<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c &gt; 0$</td>
</tr>
<tr>
<td>$\int_0^t f(t-\tau)g(\tau) , d\tau$</td>
<td>$F(s)G(s)$</td>
</tr>
<tr>
<td>$\delta(t-c)$</td>
<td>$e^{-cs}$</td>
</tr>
<tr>
<td>$f^{(n)}(t)$</td>
<td>$s^n F(s) - s^{n-1} F(0) - \cdots - F^{(n-1)}(0)$</td>
</tr>
<tr>
<td>$(-t)^n f(t)$</td>
<td>$F^{(n)}(s)$</td>
</tr>
</tbody>
</table>

(from the textbook, page 317)