20: Step functions. Differential equations with discontinuous forcing functions (sections 6.3 and 6.4)

1. Consider the n-th order linear ODE with constant coefficients:
   \[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_1 y' + a_0 y = g(t), \]
   where \( g(t) \) is a piecewise continuous function (function with jump discontinuities).

Jump discontinuities occur naturally in engineering problems such as electrical circuits with on/off switches. To handle such behavior, Heaviside introduced the following step function.

2. **Unit Step Function** \( u_c(t) \) \((c \geq 0)\) is defined by
   \[
   u_c(t) = \begin{cases} 
   0, & 0 \leq t < c \\
   1, & t \geq c 
   \end{cases}
   \]

3. When a function \( f(t) \) defined for \( t \geq 0 \) is multiplied by \( u_c(t) \), this unit step function "turns off" a portion of the graph of that function. For example, consider \((t^2 + 1)u_3(t)\).

4. **FACT 1.** Any function with jump discontinuities at \( t = c_1, c_2, \ldots, c_k \) can be represented in terms of unit step functions. In other words, we can use unit step function to write a piecewise-defined functions in a compact form.

5. Express \( f \) in terms of unit step function
   \[
   (a) \ f(t) = \begin{cases} 
   4, & 0 \leq t < 3 \\
   1, & 3 \leq t < 5 \\
   -2, & 5 \leq t 
   \end{cases}
   
   (b) \ f(t) = \begin{cases} 
   g(t), & 0 \leq t < a \\
   h(t), & a \leq t 
   \end{cases}
   
   (c) \ f(t) = \begin{cases} 
   3, & 0 \leq t < 2 \\
   1, & 2 \leq t < 3 \\
   t, & 3 \leq t < 5 \\
   t^2, & 5 \leq t 
   \end{cases}
   
6. **FACT 2.** **Translation in \( t \) property** for Laplace Transform: if \( F(s) = \mathcal{L}\{f(t)\} \) then
   \[
   F(s) = \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs} F(s).
   \]

7. Find \( \mathcal{L}\{u_c(t)\} \)

8. Duality between Laplace transform and its inverse:
   \[
   \begin{array}{|c|c|c|}
   \hline
   \text{Derivative} & \mathcal{L}\{f'(t)\} = sF(s) - f(0) & \mathcal{L}^{-1}\{F'(s)\} = -tf(t) \\
   \text{Translation} & \mathcal{L}\{e^{at}f(t)\} = F(s-a) & \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c) \\
   \hline
   \end{array}
   \]
9. Let \( f(t) \) will be the same as in 5(c). Find \( \mathcal{L}\{f\} \).

10. Find the inverse Laplace transform of

\[
H(s) = \frac{e^{-4s}}{s^2 + 9} + \frac{se^{-3s}}{s^2 + 4}
\]

11. Let

\[
g(t) = \begin{cases} 
20, & 0 \leq t < 3\pi, \\
0, & 3\pi \leq t < 4\pi \\
20, & 4\pi \leq t
\end{cases}
\]

(a) Solve IVP:

\[
y'' + 2y' + 2y = g(t), \quad y(0) = 10, \quad y'(0) = 0,
\]

Solution:

\textbf{Step 1.} Express \( g(t) \) in compact form.

\textbf{Step 2.} Find \( \mathcal{L}\{g\} = G(s) \).

\textbf{Step 3.} Find \( \mathcal{L}\{y'' + 2y' + 2y\} \).

\textbf{Step 4.} Combine steps 2\& 3 to get \( \mathcal{L}\{y(t)\} = Y(s) \).

\textbf{Step 5.} Apply inverse Laplace transform to find \( y(t) \). This step usually requires partial fraction decomposition.

(b) Sketch the graph of \( y(t) \).