28: Repeated Eigenvalues (sec. 7.5 (continued) and sec. 7.8)

Preliminary

1. **Definition** One says that \( n \) vectors \( v_1, \ldots, v_n \) constitute a basis in \( \mathbb{R}^n \) if any other vector \( v \) in \( \mathbb{R}^n \) can be uniquely represented as a linear combination of \( v_1, \ldots, v_n \), i.e. there exist constants \( c_1, \ldots, c_n \) such that

\[
v = c_1 v_1 + \ldots c_n v_n.
\]

Repeated eigenvalues

2. Recall that \( A \) is an \( n \times n \) real matrix. If \( m \) is a positive integer and \( (\lambda - \lambda_k)^m \) is a factor of the characteristic polynomial \( \det(A - \lambda I) \) while \( (\lambda - \lambda_k)^{m+1} \) is not a factor, then \( \lambda_k \) is said to be an **eigenvalue of multiplicity** \( m \). Note that \( m \leq n \).

3. Let \( A \) has \( n \) distinct real eigenvalues (i.e. all eigenvalues are of multiplicity one). In this case the corresponding eigenvectors \( v_1, \ldots, v_n \) constitute a basis in \( \mathbb{R}^n \). Recall that a fundamental set of solutions of system \( X' = AX \) in this case is \( \{e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n\} \).

Case 1: There is a Basis of Eigenvectors

4. If for a given matrix \( A \) with real eigenvalues \( \lambda_1, \ldots, \lambda_n \) (where some of them may be of multiplicity 2 or more, i.e. may be repeated several times) there exists a basis \( v_1, \ldots, v_n \) of eigenvectors then \( \{e^{\lambda_1 t}v_1, \ldots, e^{\lambda_n t}v_n\} \) is a fundamental set of solutions of system \( X' = AX \).

5. **Basis of eigenvectors does not always exist.** For example, if \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) then there is no basis of eigenvectors of this matrix.

6. Basis of eigenvectors always exists for a special kind of matrix know as a **symmetric** matrix: \( A^T = A \), or equivalently, \( a_{ij} = a_{ji} \) for all \( i, j \).

7. **Example.** Find general solution of the system

\[
\begin{align*}
  x'_1 &= 3x_1 + 2x_2 + 4x_3 \\
  x'_2 &= 2x_1 + 2x_3 \\
  x'_3 &= 4x_1 + 2x_2 + 3x_3
\end{align*}
\]

Case 2: There is NO Basis of Eigenvectors

8. Suppose that \( \lambda \) is an eigenvalue of **multiplicity two** and that there is only one eigenvector \( v \) associated with this value. In other words, there is no basis of eigenvectors of \( A \). A second
solution for a fundamental set can be found in the form

\[ te^{\lambda t}v + e^{\lambda t}w, \]

where \( w \) is so called \textit{generalized eigenvector} satisfying the condition

\[ (A - \lambda I)w = v. \]

9. Fundamental set of solutions in the case of repeated eigenvalues for \( n = 2 \):

\[ \{ e^{\lambda t}, te^{\lambda t}v + e^{\lambda t}w \} \]

10. \textbf{Example.} Consider the system:

\[
\begin{align*}
    x_1' &= -3x_1 + \frac{5}{2}x_2 \\
    x_2' &= -\frac{5}{2}x_1 + 2x_2
\end{align*}
\]

\( a) \) Find general solution of the system.

\( b) \) Find solution of the system satisfying \( x_1(0) = 2, x_2(0) = 1. \)