10.2: SERIES

A series is a sum of sequence:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

For a given sequence\(^1\) \{a_k\}_k=1^\infty define the following:

\[
\begin{align*}
S_1 &= a_1 \\
S_2 &= S_1 + a_2 = a_1 + a_2 \\
S_3 &= S_2 + a_3 = a_1 + a_2 + a_3 \\
S_4 &= S_3 + a_4 = a_1 + a_2 + a_3 + a_4 \\
\vdots &\quad \vdots \\
S_n &= S_{n-1} + a_n = \sum_{k=1}^{n} a_k \\
\end{align*}
\]
The $s_n$'s are called **partial sums** and they form a sequence $\{s_n\}_{n=1}^\infty$.

We want to consider the limit of $\{s_n\}_{n=1}^\infty$:

$$
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \sim \sum_{k=1}^{\infty} a_k
$$

If $\{s_n\}_{n=1}^\infty$ is convergent and $\lim_{n \to \infty} s_n = s$ exists as a real number, then the series $\sum_{k=1}^{\infty} a_k$ is **convergent**. The number $s$ is called the sum of the series.\(^2\)

If $\{s_n\}_{n=1}^\infty$ is divergent then the series $\sum_{k=1}^{\infty} a_k$ is **divergent**.

\(^2\)When we write $\sum_{k=1}^{\infty} a_k = s$ we mean that by adding sufficiently many terms of the series we can get as close as we like to the number $s$. 
GEOMETRIC SERIES

\[ a + ar + ar^2 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n \quad (a \neq 0) \]

Each term is obtained from the preceding one by multiplying it by the common ratio \( r \).

**FACT:** The geometric series is convergent if \( |r| < 1 \) and its sum is

\[ \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}. \]

If \( |r| \geq 1 \), the geometric series is divergent.
EXAMPLE 1. Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.

(a) \( \sum_{n=1}^{\infty} 5 \cdot \left( \frac{2}{7} \right)^{n-1} = \sum_{n=1}^{\infty} \frac{10}{7} \cdot \left( \frac{2}{7} \right)^{n-1} \)

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2}{7} \]

\[ s = \frac{a}{1-r} = \frac{10/7}{1-2/7} = \frac{10}{5} = 2 \]

(b) \( \sum_{n=0}^{\infty} \frac{(-4)^3}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{(-64)}{5^n} \)

\[ s = \sum_{n=0}^{\infty} \left( \frac{-64}{5} \right)^n \]

\[ r = -\frac{64}{5} < -1 \]

(c) \( 1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \ldots = \sum_{n=0}^{\infty} \left( \frac{3}{2} \right)^n \)

\[ a = 1, \quad r = -\frac{3}{2} < -1 \]

\[ \text{geometric series divergent} \]
\[
\sum_{n=1}^{\infty} 4^{n+1} \cdot 9^{2-n} = \sum_{n=1}^{8} \frac{4^{n+1} \cdot 9^{-(n-2)}}{9^{n-2}} = \sum_{n=1}^{8} \frac{4^{n+1}}{9^{n-1}} = \sum_{n=1}^{8} \frac{4^{n+1}}{9^{n-1} \cdot 9} = \sum_{n=1}^{8} \frac{4^{n+1}}{9^{n-1}}
\]

Geometric: \( a = 144 \)
\( r = \frac{4}{9} \)
\( S = \frac{a}{1-r} = \frac{144}{1-\frac{4}{9}} = \frac{1296}{5} \)
EXAMPLE 2. Write the number \( .\overline{17} \) as a ratio of integers.

\[
.\overline{17} = \frac{m}{n}
\]

\[
.\overline{17} = .171717171717\ldots = .17 + .0017 + .000017 + \ldots
\]

\[
= .17 + .0017 \cdot 10^{-2} + .000017 \cdot 10^{-4} + \ldots
\]

\[
= .17 \cdot 10^{-2} + .0017 \cdot 10^{-4} + \ldots
\]

Geometric series

with \( a = .17 \)

\( r = 10^{-2} = .01 \)

Convergent \( \frac{a}{1-r} \) \( |r| < 1 \)

\[
.\overline{17} = \frac{a}{1-r} = \frac{.17}{1-0.01} = \frac{17}{99}
\]
TELESCOPING SUM

Let \( b_n \) be a given sequence. Consider the following series:

\[
\sum_{n=1}^{\infty} (b_n - b_{n+1})
\]

\[
a_n = b_n - b_{n+1}
\]

Partial sum

\[
S_n = a_1 + a_2 + \ldots + a_n = b_1 - b_2 + b_2 - b_3 + \ldots + b_{n-1} - b_n + b_n - b_{n+1}
\]

\[
= b_1 - b_{n+1}
\]

The sum of telescoping series (if it converges)

\[
S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} (b_1 - b_{n+1})
\]

\[
S = b_1 - \lim_{n \to \infty} b_{n+1}
\]

Question: Are these series telescoping?

\[
\sum_{n=1}^{\infty} b_{n+1} - b_n
\]

\[
\sum_{n=1}^{\infty} b_{n+2} - b_n
\]

\[
\sum_{n=1}^{\infty} b_n - b_{n+3}
\]
EXAMPLE 3. Determine whether the following series converges or diverges. If it is converges, find the sum. If it is diverges, explain why.

(a) \( \sum_{n=1}^{\infty} \left( \sin \frac{1}{n} - \sin \frac{1}{n+1} \right) = \sum_{n=1}^{\infty} b_n - b_{n+1} \)  \[ \text{Telescoping} \]

\[ b_n = \sin \frac{1}{n} \]

\[ S_n = b_1 - \lim_{n \to \infty} b_{n+1} = \sin 1 - \lim_{n \to \infty} \sin \frac{1}{n+1} \]

\[ \text{convergent and } S = \sin 1 \]

(b) \( \sum_{n=1}^{\infty} \ln \frac{n+1}{n+2} = \sum_{n=1}^{\infty} \frac{\ln (n+1) - \ln (n+2)}{b_n} \)  \[ \text{Telescoping} \]

\[ \phi = b_1 - \lim_{n \to \infty} b_{n+1} = \ln 2 - \lim_{n \to \infty} \ln (n+2) = -\infty \]

(c) \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \)

Use partial fraction decomposition. \( \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1} \)

\[ l = A(n+1) + Bn \]

\[ n = -1 \Rightarrow B = 1 \]

\[ n = 0 \Rightarrow A = 1 \]  \[ \text{Telescoping} \]

\[ S = \lim_{n \to \infty} S_n = b_1 - \lim_{n \to \infty} b_{n+1} = 1 - \lim_{n \to \infty} \frac{1}{n+1} = \boxed{1} \]

\[ \text{sum convergent} \]
THEOREM 4. If the series \( \sum_{n=1}^{\infty} a_n \) is convergent, then \( \lim_{n \to \infty} a_n = 0 \).

REMARK 5. The converse is not necessarily true.

THE TEST FOR DIVERGENCE:
If \( \lim_{n \to \infty} a_n \) does not exist or if \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

REMARK 6. If you find that \( \lim_{n \to \infty} a_n = 0 \) then the Divergence Test fails and thus another test must be applied.
EXAMPLE 7. Use the test for Divergence to determine whether the series diverges.

(a) \[ \sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)} \]

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{3(n+1)(n+2)} = \frac{1}{3} \neq 0 \]

The series diverges

(b) \[ \sum_{n=1}^{\infty} \cos \frac{\pi n}{2} \]

\[ \lim_{n \to \infty} \cos \frac{\pi n}{2} \text{ DNE because } \]

n is odd \( \Rightarrow \) \( \cos \frac{\pi n}{2} \neq 0 \)

n is even \( \Rightarrow \) \( \cos \frac{\pi n}{2} = \pm 1 \) (oscillating)

The series diverges

(c) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \]

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0 \]

\( \therefore \)

\[ \lim_{n \to \infty} a_n = 0 \]

Divergence Test Fails here. Thus, to make a conclusion we have to use some other test. (See Next Sections)