10.5: Power Series

DEFINITION 1. A power series about $x = a$ (or centered at $x = a$), or just power series, is any series that can be written in the form

$$
\sum_{n=0}^{\infty} c_n (x - a)^n,
$$

where $a$ and $c_n$ are numbers. The $c_n$'s are called the coefficients of the power series.

For example

$$
\sum_{n=0}^{\infty} x^n \quad \text{then center } a = 0
$$

coeff. $c_n = 1$

general term $a_n = x^n$

THEOREM 2. For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ there are only 3 possibilities:

1. The series converges only for $x = a$.

2. The series converges for all $x$.

3. There is $R > 0$ such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. We call such $R$ the radius of convergence.

REMARK 3. In case 1 of the theorem we say that $R = 0$ and in case 2 we say that $R = \infty$

$$
|x - a| < R \implies -R < x - a < R \\
\implies a - R < x < a + R
$$

DEFINITION 4. An interval of convergence is the interval of all $x$'s for which the power series converges.

Points $x = a \pm R$ are called end points

and should be investigated separately
EXAMPLE 5. Assume that it is known that the series $\sum_{n=0}^{\infty} c_n(x + 3)^n$ converges when $x = -10$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series:

- $\sum_{n=0}^{\infty} c_n 2^n$ converges when $x = 3$ (see diagram below)
- $\sum_{n=0}^{\infty} c_n (-11)^n$ diverges when $x = -14$
- $\sum_{n=0}^{\infty} c_n 8^n$ diverges when $x = 5$
- Not enough information.
**Ex 0.**

$$\sum_{n=0}^{\infty} \frac{x^n}{a_n}$$

Geometric series with $r = x$

$\Rightarrow$ converges $|r| < 1$, or

$|x| < 1$

Interval of conv. $[-1 < x < 1]$

$R = 1$

Apply Ratio Test

$$L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n \to \infty} \frac{|x|^{n+1}}{|x|^n} = \lim_{n \to \infty} |x| = |x| < 1$$

$R = 1$. 

$\therefore$
EXAMPLE 6. Given \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n \).

(a) Find the radius of convergence.

Apply Ratio Test:
\[
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|(x+3)^{n+1}|}{|x+3|^n} = \frac{|x+3|}{4} < 1 \implies |x+3| < 4
\]

Hence, the radius of convergence is \( R = 4 \).

(b) Find the interval of convergence.

\[
L = \frac{|x+3|}{4} < 1 \quad \text{and} \quad L = 1 \quad \text{(end points)}
\]

By (a) \( L = 1 \Rightarrow |x+3| = 4 \)  
\[x+3 = \pm 4\]

Plug in \( x = 3 \) (i.e. \( x = 1 \))

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} 4^n = \sum_{n=1}^{\infty} (-1)^n n
\]

Diverges by Divergent Test:

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n n \quad \text{DNG}
\]

Plug in second end point where \( x = -1 \) (i.e. \( x = -7 \))

\[
\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (-4)^n = \sum_{n=1}^{\infty} (-1)^n n (-1)^n 4^n
\]

\[= \sum_{n=1}^{\infty} n \quad \text{by DT, } \lim_{n \to \infty} a_n = \infty.
\]

Conclusion: No convergence at end points and then \((-7, 1)\) is interval of convergence.
EXAMPLE 7. Given \( \sum_{n=1}^{\infty} \frac{2^n}{n} (3x - 6)^n = \frac{6}{n} (x-2)^n \)

(a) Find the radius of convergence.

\[
|a_n| = \frac{6^n}{n} (x-2)^n
\]

\[
|a_{n+1}| = \frac{6^{n+1}}{n+1} (x-2)^{n+1}
\]

Apply Ratio Test

\[
L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{6^{n+1}}{n+1} \frac{|x-2|^{n+1}}{6^n |x-2|^n}
\]

\[
= 6|x-2| \lim_{n \to \infty} \frac{n}{n+1} = 6|x-2| < 1
\]

\[
|x-2| < \frac{1}{6}
\]

\[
R = \frac{1}{6}
\]
Find the interval of convergence.

\[ L < 1 \quad \& \quad \text{inspect} \quad L = 1 \quad \text{(end points)} \]

\[ 1x - 2 < \frac{1}{6} \]

\[-\frac{1}{6} < x - 2 < \frac{1}{6} \]

\[-\frac{1}{6} + 2 < x - 2 + 2 < \frac{1}{6} + 2 \]

\[ \frac{11}{6} < x < \frac{13}{6} \]

Plug in

\[ \sum_{n=1}^{\infty} \frac{6^n}{n} \left( \frac{1}{6} \right)^n \]

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

Divergent as p-series \( p = 1 \)

(Or use Integral Test)

Converges by AST

\( \lim_{n \to \infty} \frac{1}{n} = 0 \)

and \( \{ \frac{11}{6}, \frac{13}{6} \} \)

Conclusion: The interval of convergence is \( \left[ \frac{11}{6}, \frac{13}{6} \right] \)

\[ \text{OR} \quad \left( \frac{11}{6}, \frac{13}{6} \right) \]
EXAMPLE 8. Given \( \sum_{n=1}^{\infty} \frac{(-1)^n}{(3n+1)!} (x+8)^n \).

(a) Find the radius of convergence.

\[
|a_n| = \left| \frac{(-1)^n (x+8)^n}{(3n+1)!} \right| = \frac{|x+8|^n}{(3n+1)!}
\]

\[
|a_{n+1}| = \frac{|x+8|^{n+1}}{(3(n+1)+1)!} = \frac{|x+8|^{n+1}}{(3n+4)!}
\]

Apply Ratio Test

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x+8|^{n+1}}{(3n+4)!} \cdot \frac{(3n+1)!}{|x+8|^n} = |x+8| \cdot \lim_{n \to \infty} \frac{(3n+1)!}{(3n+4)!} = 0 < 1
\]

\[
R = \infty
\]

(b) Find the interval of convergence.

\((-\infty, \infty)\)
EXAMPLE 9. Given \( \sum_{n=1}^{\infty} \frac{(2n)!}{q^{n-1}}(x+8)^n \).

(a) Find the radius of convergence.

\[
|a_{n+1}| = \frac{(2(n+1))!}{q^{n+1}-1} \frac{|x+8|^{n+1}}{(2n+2)!} \leq \frac{(2n+2)!}{q^n} \frac{|x+8|^{n+1}}{q^{n+1}}
\]

Apply Ratio Test

\[
L = \lim_{n \to \infty} \frac{|x+8|}{q} = \frac{\lim_{n \to \infty} (2n+2)!}{q^n} \frac{|x+8|}{(2n)!} = \left\{ \begin{array}{ll}
\infty > 1, & x \neq -8 \\
0 < 1, & x = -8
\end{array} \right.
\]

The series converges at \( x = -8 \) only

\[
\Rightarrow R = 0
\]

(b) Find the interval of convergence.

= singleton \( \{ -8 \} \)
EXAMPLE 10. Given \( \sum_{n=1}^{\infty} \frac{x^{5n+5}}{3^{n+1}(n+1)} \), power series

(a) Find the radius of convergence.

\[
|a_n| = \frac{|x|^{5n+5}}{3^{n+1}(n+1)}
\]

\[
|a_{n+1}| = \frac{|x|^{5(n+1)+5}}{3^{(n+1)+1}(n+2)} = \frac{|x|^{5n+10}}{3^{n+2}(n+2)}
\]

Apply Ratio Test

\[
L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x|^{5n+10}}{3^{n+2}(n+2)} \cdot \frac{3^{n+1}(n+1)}{|x|^{5n+5}}
\]

\[
= \frac{|x|^5}{3} \lim_{n \to \infty} \frac{n+1}{n+2} = \frac{|x|^5}{3} < 1
\]

\[
|x|^5 < 3 \quad \Rightarrow \quad |x| < \sqrt[5]{3}
\]

\[
\Rightarrow R = \sqrt[5]{3}
\]
(b) Find the interval of convergence.

\[ L = 1 \implies |x| = \frac{5}{3} \iff x = \pm \frac{5}{\sqrt{3}} \]

Plug in to

\[ \sum_{n=1}^{\infty} \frac{(x^5)^{n+1}}{3^{n+1}(n+1)} \]

For

\[ x = -\frac{5}{\sqrt{3}} \]

\[ \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{3^{n+1}(n+1)} \]

converges by AST:

\[ \lim_{n \to \infty} \frac{1}{n+1} = 0 \]

\[ \{ \frac{1}{n+1} \} \downarrow \]

For

\[ x = \frac{5}{\sqrt{3}} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n+1} \]

diverges by integral test:

\[ \int_{1}^{\infty} \frac{dx}{x+1} = \infty \]

Conclusion: The interval of convergence is

\[ -\frac{5}{\sqrt{3}} \leq x < \frac{5}{\sqrt{3}} \]

or \( [-\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}) \).