7.1: Areas Between Curves

One of interpretations of definite integral

\[
\text{Area}(D) = \int_a^b f(x) \, dx, \quad f(x) \geq 0 \quad \text{on} \quad [a, b]
\]

is the area between the graph of \( y = f(x) \) and the \( x \)-axis on \( [a, b] \).

\[
D = \left\{ (x, y) \mid a \leq x \leq b, \quad 0 \leq y \leq f(x) \right\}
\]
For example, if \( f(x) = \cos x \) and \( x \in [0, \frac{\pi}{2}] \) then

\[
\text{Area} = \int_0^{\frac{\pi}{2}} \cos x \, dx = \sin x \bigg|_0^{\frac{\pi}{2}} = 1
\]

If \( f(x) \geq 0 \) on \([a, b]\) then \( \int_a^b f(x) \, dx \geq 0 \)

If \( f(x) \leq 0 \) on \([a, b]\) then \( \int_a^b f(x) \, dx \leq 0 \)

The previous example on \([0, \frac{2\pi}{3}]\):

\[
\int_0^{\frac{2\pi}{3}} \cos x \, dx 
eq \text{Area} \int_0^{\frac{\pi}{2}} \cos x \, dx + \left( -\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos x \, dx \right)
\]
Our goal: Find the area between two curves.

CASE I. Determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ assuming $f(x) \geq g(x)$ on $[a, b]$.

In other words, find the area of the region $D$ defined by

$$D = \{(x, y) \mid a \leq x \leq b, \ g(x) \leq y \leq f(x)\}$$

Solution:

$$A = A(D) = \int_a^b [f(x) - g(x)] \, dx$$

Explanation:
CASE II. Determine the area between \( x = f(y) \) and \( x = g(y) \) on the interval \([c, d]\) assuming \( f(y) \geq g(y) \) on \([c, d]\).

In other words, find the area of the region \( D \) defined by

\[
\begin{align*}
    y & \quad x = g(y) \\
    a & \quad f(y) \\
     c & \quad \text{Domain} \\
     0 & \quad \text{Range}
\end{align*}
\]

**Solution:**

\[
A = A(D) = \int_{c}^{d} f(y) - g(y) \, dy
\]

The above formulas in the "word" form:

**CASE I** \( A = \int_{a}^{b} \left( \text{upper function} \right)(x) - \left( \text{lower function} \right)(x) \, dx \)

**CASE II** \( A = \int_{c}^{d} \left( \text{right function} \right)(y) - \left( \text{left function} \right)(y) \, dy \)
Coming back to the previous example: \( f(x) = \cos x \), where \( 0 \leq x \leq 2\pi/3 \) we get:

\[
A = \int_0^{\pi/2} (\cos x - 0) \, dx + \int_{2\pi/3}^{\pi/2} (0 - \cos x) \, dx = \ldots
\]
EXAMPLE 1. Determine the area of the region enclosed (bounded by) by \( y = x^2 \) and \( y = \sqrt{x} \).

\[
\int_0^1 (\sqrt{x} - x^2) \, dx = \frac{x^{3/2}}{3/2} - \frac{x^3}{3} \bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\]

REMARK 2.  
1. The limits of integration in the above example were determined as the intersection points of the two curves.

2. Sketch of a graph of the region is recommended (it helps to determine which of the functions is upper/right).

3. The area between two curves will always be positive.
EXAMPLE 3. Determine the area of the region bounded by $y = \frac{1}{x}$ and $y = -1$, $x = 1$, $x = 3$

\[
\text{Area} = \int_{1}^{3} \left( \frac{1}{x} - (-1) \right) \, dx = \left. \ln |x| + x \right|_{1}^{3} = \\
= \ln 3 - \ln 1 + 3 - 1 = \\
= \ln 3 + 2
\]
EXAMPLE 4. Determine the area of the region bounded by $y = 2x^2 + 4$ and $y = 4x + 10$.

\[
A = \int_{a}^{b} (V_F) - (L_F) \, dx = \int_{a}^{b} (4x+10) - (2x^2+4) \, dx
\]

Find $x$-coord. of intersection points

\[
\begin{align*}
2x^2 + 4 &= 4x + 10 \\
x^2 + 2 &= 2x + 5 \\
x^2 - 2x - 3 &= 0 \\
(x+1)(x-3) &= 0
\end{align*}
\]

$x = -1$ or $x = 3$

$A = \frac{64}{3}$
EXAMPLE 5. Determine the area of the region bounded by \( y = 2x^2 + 4 \) and \( y = 4x + 10 \), \( x = -2 \), \( x = 5 \).

As in Ex. 4

\[
A = \frac{64}{3}
\]

\[
A = \int_{-2}^{5} (U_F) - (L_F) \, dx = \int_{-2}^{5} 2x^2 + 4 - (4x + 10) \, dx + \frac{64}{3} + \int_{3}^{5} 2x^2 + 4 - (4x + 10) \, dx
\]

\[
\Rightarrow \int_{-2}^{-1} 2x^2 + 4 - (4x + 10) \, dx + \frac{64}{3} + \int_{3}^{5} 2x^2 + 4 - (4x + 10) \, dx = \frac{142}{3}
\]
EXAMPLE 6. Determine the area of the region enclosed by \( y = \sin x, \ y = \sin 2x, \ x = 0, \ x = \pi/2 \).

Find \( C \):

\[
\sin 2x = \sin x \\
2 \sin x \cos x = \sin x \\
2 \sin x \cos x - \sin x = 0 \\
\sin x (2 \cos x - 1) = 0 \\
\sin x = 0 \text{ or } 2 \cos x = 1 \\
x = 0 \text{ or } \cos x = \frac{1}{2} \\
x = \frac{\pi}{3} = C
\]

\[
A = \int_0^{\pi/2} (UF) - (LF) \, dx = \int_0^{\pi/3} \sin 2x - \sin x \, dx + \int_{\pi/3}^{\pi/2} \sin x - \sin 2x \, dx
\]

\[
= \left( -\frac{1}{2} \cos 2x - (-\cos x) \right) \bigg|_0^{\pi/3} + \left( -\cos x - \left(-\frac{1}{2} \cos 2x\right) \right) \bigg|_{\pi/3}^{\pi/2}
\]

\[
= \ldots = \frac{1}{2}
\]
EXAMPLE 7. Determine the area of the region enclosed by $x = \frac{1}{2}y^2 - 3$, $y = x - 1$.

$$A = \int_{c}^{d} (RF) - (LF) \, dy = \int_{c}^{d} (y + 1) - (\frac{1}{2}y^2 - 3) \, dy =$$

Find $c$ and $d$:

$\frac{1}{2}y^2 - 3 = y + 1 \quad (x 2)$

$y^2 - 6 = 2y + 2$

$y^2 - 2y - 8 = 0$

$(y - 4)(y + 2) = 0$

$y = 4 \quad \text{or} \quad y = -2 = c$

$y = 4 \quad \text{or} \quad y = -2 = d$

$$= \int_{-2}^{4} (y + 4 - \frac{1}{2}y^2) \, dy =$$

$$= \ldots = 18$$
EXAMPLE 8. Determine the area of the region bounded by the $x$-axis, the curve $y = x^2$ and tangent line to this curve at the point $(1, 1)$.

\[
A = \int_0^1 (RF) - (LF) dy =\]

\[
= \int_0^1 \left( \frac{y^2}{4} + \frac{y}{2} - \frac{2}{3} \right) dy =\]

\[
= \left. \frac{y^2}{4} + \frac{y}{2} - \frac{2}{3} \right|_0^1 =\]

\[
= \frac{1}{4} + \frac{1}{2} - \frac{2}{3} = \frac{1}{12}\]

Tangent: $y = mx + b$

\[
m = (x^2)'\bigg|_{x=1} = 2\]

\[
y = 2x + b \quad (1, 1) : \quad 1 = 2 + b \Rightarrow b = -1\]

\[
y = 2x - 1\]

\[
x = \frac{y + 1}{2} (RF)\]