14: Nonhomogeneous Equations. Method of Undetermined Coefficients (section 3.5)

1. If $y_1(t)$ and $y_2(t)$ are two solutions of a second-order nonhomogeneous linear ODE,

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \neq 0.$$

Then $y_1(t) - y_2(t)$ is a particular solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0.$$

2. THEOREM: Solution of Nonhomogeneous Linear Equation

Let

$$y'' + p(t)y' + q(t)y = g(t), \quad g(t) \neq 0$$

be a second-order nonhomogeneous linear differential equation. If $y_p(t)$ is a particular solution of this equation and $y_h(t)$ is the general solution of the corresponding homogeneous equation,

$$y'' + p(t)y' + q(t)y = 0,$$

then

$$y(t) = y_h(t) + y_p(t)$$

is the general solution of the nonhomogeneous equation.
3. To solve a nonhomogeneous linear ODE:

_Step 1:_ Find a particular solution of a nonhomogeneous linear ODE. \( y_p(t) \)

_Step 2:_ Find general solution of the corresponding homogeneous linear ODE. \( y_h(t) \)

_Step 3:_ Add the results of Steps 1&2. \( y = y_p + y_h \)

4. One solution of \( y'' - y = t \) is \( y(t) = -t \), as you can verify. What is the general solution?

\[
\begin{align*}
y'' - y &= 0 \\
0 - 1 &= 0 \\
r^2 - 1 &= 0 \\
r &= \pm 1 \\
y_h(t) &= C_1 e^t + C_2 e^{-t}
\end{align*}
\]

\[
y(t) = y_p + y_h
\]

\[
y(t) = -t + C_1 e^t + C_2 e^{-t}
\]

5. Consider

\[ y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) \]  \hspace{1cm} (1)

If \( g(t) = y_p(t) \) and \( y(t) = Y_p(t) \) are particular solutions of

\[ y'' + p(t)y' + q(t)y = g_1(t) \]

and

\[ y'' + p(t)y' + q(t)y = g_2(t). \]

respectively, then \( y(t) = y_p(t) + Y_p(t) \) is a particular solution of (1).
Method of Undetermined Coefficients

6. Consider a particular class of nonhomogeneous linear ODE with constant coefficients

$$ay'' + by' + cy = g(t),$$

where $a, b, c$ are real constants and $g(t)$ involves linear combinations, sums and products of

$$t^m, \quad e^{\alpha t}, \quad \sin(\beta t), \quad \cos(\beta t).$$

7. The idea of the Method of Undetermined Coefficients: guess a particular solution $y_p(t)$ using a generalized form of $g(t)$. Then by substitution determine the coefficients for the generalized solution.

8. Find general solution:

(a) $y'' - 3y' + 2y = 4e^{3t}$

Homogeneous

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r_1 = 1, \quad r_2 = 2$

$y_h = c_1 e^t + c_2 e^{2t}$

Undetermined Coefficient

$y_p(t) = Ae^{3t}$

$y'' + 3y' + 2y = 4e^{3t}$

$y_p(t) = 2e^{3t}$

General solution

$y(t) = 2e^{3t} + c_1 e^t + c_2 e^{2t}$

$y_p$ \hspace{1cm} $y_h$
(b) \( y'' - 3y' + 2y = 4e^t \)

\[ y_h = c_1 e^t + c_2 e^{2t} \]

\[ y_p = -4 t e^t \]

\[ y(t) = -4 t e^t + c_1 e^t + c_2 e^{2t} \]

\[ 2 \begin{align*} \alpha &= 1 \\ r_1 &= 1, r_2 &= 2 \end{align*} \Rightarrow s = 1 \]

\[ 4 e^t = -3 A e^t + 2 A e^t \]

\[ 4 e^t = -A e^t \]

\[ A = -4 \]

(c) \( y'' + 10y' + 25y = 3e^{-5t} \)

\[ r^2 + 10r + 25 = 0 \]

\[ (r + 5)^2 = 0 \]

\[ r_1 = r_2 = -5 \]

\[ y_h(t) = C_1 e^{-5t} + C_2 t e^{-5t} \]

\[ \alpha = -5 \]

\[ r_1 = r_2 = -5 \]

\[ s = 2 \]

\[ y_p(t) = \frac{3}{2} t^2 e^{-5t} \]

\[ y(t) = \frac{3}{2} t^2 e^{-5t} + C_1 e^{-5t} + C_2 t e^{-5t} \]
\[ g(t) = \left( \text{polynomial}(t) \right) e^{\alpha t} \cos(\beta t) \quad \text{or} \quad e^{\alpha t} \sin(\beta t) \]

9. Multiplicity \( s \) of a given number \( \alpha + i\beta \)

If \( g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) e^{\alpha t}, \quad \beta = 0 \), then

- \( \alpha \) doesn’t coincide with a root of characteristic polynomial \( \Rightarrow s = 0 \).
- \( \alpha \) coincides with a non repeated root of characteristic polynomial \( \Rightarrow s = 1 \).
- \( \alpha \) coincides with a repeated root of characteristic polynomial \( \Rightarrow s = 2 \).

If \( g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_1 t^{n-1} + \ldots + B_n) e^{\alpha t} \cos(\beta t) \) or \( g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) e^{\alpha t} \sin(\beta t)), \quad \beta \neq 0 \), then

- \( \alpha + i\beta \) doesn’t coincide with a root of characteristic polynomial \( \Rightarrow s = 0 \).
- \( \alpha + i\beta \) coincides with a (complex) root of characteristic polynomial \( \Rightarrow s = 1 \).

10. If \( g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) e^{\alpha t} \cos(\beta t) \) or \( g(t) = (B_0 t^n + B_1 t^{n-1} + \ldots + B_n) e^{\alpha t} \sin(\beta t) \) then we choose a particular solution in the form

\[ y_p(t) = t^s (A_0 t^n + A_1 t^{n-1} + \ldots + A_n) e^{\alpha t} \cos(\beta t) \]

or

\[ y_p(t) = t^s (A_0 t^n + A_1 t^{n-1} + \ldots + A_n) e^{\alpha t} \sin(\beta t), \]

respectively.
11. Find general solution of

\[ y'' + 2y' + 5y = 3 \sin 2t \]
12. Determine a suitable choice for $y_p$:

$$e^{at} \cos bt / e^{at} \sin bt$$

<table>
<thead>
<tr>
<th>$ay'' + by' + cy = g(t)$</th>
<th>$r_1, r_2$</th>
<th>$\alpha + i\beta$</th>
<th>$s$</th>
<th>$y_p(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y'' = e^t$</td>
<td>$r_1 = r_2 = 0$</td>
<td>$\alpha = 1, \beta = 0$</td>
<td>$0$</td>
<td>$Ae^t (t^5 = 1)$</td>
</tr>
<tr>
<td>$y'' - 4y' + 4 = e^{2t}$</td>
<td>$r_1 = r_2 = 2$</td>
<td>$\alpha = 2, \beta = 2$</td>
<td>$2$</td>
<td>$Ae^t e^{2t}$</td>
</tr>
<tr>
<td>$y'' + 2y' + 10y = 6 \sin(3t)$</td>
<td>$r_{1,2} = -1 \pm 3i$</td>
<td>$\alpha = 0, \beta = 0$</td>
<td>$0$</td>
<td>$A \sin 3t + B \cos 3t$</td>
</tr>
<tr>
<td>$y'' + 2y' + 10y = 6e^{-t} \sin(3t)$</td>
<td>$r_{1,2} = -1 \pm 3i$</td>
<td>$\alpha = 1, \beta = 1$</td>
<td>$1$</td>
<td>$Ae^t \sin 3t + Be^{-t} \cos 3t$</td>
</tr>
<tr>
<td>$y'' + 2y' + 10y = (t^3 - 1) \sin(3t)$</td>
<td>$r_{1,2} = -1 \pm 3i$</td>
<td>$\alpha = 3i, \beta = 0$</td>
<td>$0$</td>
<td>$Ae^t (t^5 + Bt^3 + C) \sin 3t + Be^{-t} (t^5 + D) \cos 3t$</td>
</tr>
<tr>
<td>$y'' + 2y' + 10y = t^2 e^{-t} \sin(3t)$</td>
<td>$r_{1,2} = -1 \pm 3i$</td>
<td>$\alpha = 1, \beta = 3i$</td>
<td>$1$</td>
<td>$Ae^t + B$</td>
</tr>
<tr>
<td>$y'' = 2t - 2013$</td>
<td>$r_1 = r_2 = 0$</td>
<td>$\alpha = 0, \beta = 2$</td>
<td>$2$</td>
<td>$(A + B) t^2$</td>
</tr>
</tbody>
</table>