16: Mechanical and Electrical Vibrations (section 3.7)

Consider linear dynamical system in which mathematical model is the following IVP:

\[ au'' + bu' + cu = g(t), \quad u(0) = u_0, \quad u'(0) = v_0. \]

Here \( g(t) \) is forcing function of the system. A solution \( u(t) \) of the DE on an interval containing \( t = 0 \) that satisfies the initial conditions is called the response of the system.

**Spring/mass systems: Free Undamped Vibration (or simple harmonic motion)**

1. A flexible spring is suspended vertically from a rigid support and the mass \( m \) is attached to the end. By Hooke’s Law, the spring itself exerts a restoring force \( F \) opposite to the direction of elongation and proportional to the amount of elongation \( L \): \( F = -kL \), where \( k \) is called the spring constant.

2. The mass \( m \) stretches the spring by \( L \) and attains a position of equilibrium, i.e. weight, \( mg \), is balanced by the restoring force:

\[ mg - kL = 0. \]

3. If the mass is displaced by an amount \( u \) from its equilibrium position, the restoring force is then \(-k(u + L)\). Free motion (i.e. no other external/retarding forces acting on the moving mass): use Newton’s second Law with the net (or resultant) force:

\[ mu'' = -k(u + L) + mg = -ku. \]
4. DE of Free Undamped Motion:

\[ u'' + \omega_0^2 u = 0, \]

where

\[ k^2 + \omega_0^2 = 0 \]

\[ \omega_0^2 = \frac{k}{m} \Rightarrow \omega_0 = \sqrt{\frac{k}{m}} \]

Initial conditions: \( u(0) = u_0, \ u'(0) = v_0 \), where \( u_0 \) is the initial displacement and \( v_0 \) is the initial velocity. For example, \( u_0 < 0 \) and \( v_1 = 0 \) mean that the mass is released from rest from a point \( |u_0| \) units above the equilibrium position.

General solution of (1) is

\[ u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = R \cos(\omega_0 t - \delta), \]

where

- \( R = \sqrt{C_1^2 + C_2^2} \) is called the amplitude of the motion.
- \( \delta \) is called the phase, or phase angle, and measures the displacement of the wave from its normal position corresponding to \( \delta = 0 \). Recall that

\[ \cos \delta = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \frac{C_1}{R}, \quad \sin \delta = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \frac{C_2}{R}. \]

- \( T = \frac{2\pi}{\omega_0} \) is the period of the motion. The number \( T \) is time it takes the mass to execute one cycle of motion (the length of the interval between two successive maxima (or minima) of \( u(t) \)).
- \( \omega_0 = \sqrt{\frac{k}{m}} \) is the natural frequency of the system.
- The frequency of motion \( f = \frac{1}{T} = \frac{\omega_0}{2\pi} \).
5. A mass weighing 4 lb stretches a spring 6 inches. At \( t = 0 \) the mass released from a point 8 inches below the equilibrium with an upward velocity of 2/3 ft/s. Determine the amplitude of vibrations, phase angle, period, natural frequency of the system and frequency of motion.

\[
\begin{align*}
mg &= 4 \Rightarrow m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8} \text{ slugs} \\
L &= 6 \text{ in} = \frac{6}{12} = \frac{1}{2} \text{ ft} \\
u(0) &= 8 \text{ in} = \frac{8}{12} = \frac{2}{3} \text{ ft} \\
u'(0) &= -\frac{2}{3} \text{ ft/s}
\end{align*}
\]

\[R, \delta, T, \omega_0, f\]

\[
\begin{align*}
mg &= kL \Rightarrow k = \frac{mg}{L} = \frac{4}{\frac{1}{2}} = 8 \\
\omega_0 &= \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{\frac{1}{8}}} = \sqrt{64} = 8 \\
T &= \frac{2\pi}{\omega_0} = \frac{2\pi}{8} = \frac{\pi}{4}
\end{align*}
\]

\[u'' + \omega_0^2 u = 0 \Rightarrow u'' + 64u = 0 \\
y^2 = -64
\]

\[r = \pm 8i
\]

General solution: \( u(t) = C_1 \cos 8t + C_2 \sin 8t \)

To find \( C_1, C_2 \) use initial conditions \( u'(t) = -8C_1 \sin 8t + 8C_2 \cos 8t \)

\[\begin{align*}
\frac{2}{3} = u(0) &= C_1 \\
\Rightarrow C_1 &= \frac{2}{3} \\
-\frac{2}{3} = u'(0) &= 8C_2
\end{align*}
\]

\[\begin{align*}
C_2 &= -\frac{2}{3 \cdot 8} = -\frac{1}{12} = C_2 \\
C_1 &= \frac{2}{3}
\end{align*}
\]

\[R = \sqrt{C_1^2 + C_2^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{12}\right)^2} = \frac{\sqrt{64 + 1}}{12} = \frac{\sqrt{65}}{12}
\]

\[\delta = \arctan\left(-\frac{1}{8}\right)
\]

\[\frac{2\pi}{2} < \delta < \frac{3\pi}{2}
\]

Graphical representation of \( C_2 \) and \( C_1 \)
Spring/mass systems: Free Damped Vibrations.

6. Assume that the mass is suspended in a viscous medium or connected to a dashpot damping device. Dampers work to counteract any movement: damping force $= -\gamma v = -\gamma u'$, where $\gamma$ is a positive damping constant.

$F_d = -\gamma u = -\gamma u'$

$$m\ddot{u} = mg - k(L+u) - \gamma u'$$

$m\ddot{u} + \gamma u' + ku = 0$
7. DE of Free Damped Motion:

\[ mu'' + \gamma u' + ku = 0. \]  \hspace{1cm} (2)

8. Discriminant of the characteristic equation \( mr^2 + \gamma r + k = 0 \) is

\[ D = \gamma^2 - 4mk. \]

**CASE 1: (Underdamping) \( D < 0 \), i.e. the roots are complex conjugate:**

\[ r_{1,2} = -\frac{\gamma}{2m} \pm i \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} =: \lambda \pm i \mu \]

\[ \lambda = -\frac{\gamma}{2m} < 0 \]

\[ u(t) = e^{-\lambda t} (C_1 \cos \mu t + C_2 \sin \mu t) \]

General solution of (2) is not periodic:

\[ u(t) = C_1 e^{-\lambda t} \cos(\mu t) + C_2 e^{-\lambda t} \sin(\mu t) = Re^{-\lambda t} \cos(\mu t - \delta), \]

where

- \( Re^{-\lambda t} \) is **damped amplitude** of vibrations
- \( \mu = \sqrt{\omega_0^2 - \frac{\gamma^2}{4m^2}} = \sqrt{\omega_0^2 - \lambda^2} \) is the **quasi frequency**
- \( T_d = \frac{2\pi}{\mu} = \frac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} \) is the **quasi period**, i.e. the time interval between two successive maxima of \( u(t) \).

Note that as \( \gamma \) increases, the quasi frequency \( \mu \) becomes smaller and the quasi period becomes bigger.
CASE 2: (Critical Damping) \( D = 0 \) (two repeated (equal) roots) In this case any slight decrease of the damping force would result in oscillatory motion. The general solution of (2) is

\[ x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} = e^{-\lambda t}(C_1 + C_2 t). \]

\[ D = \delta^2 - 4mk = 0 \]

\[ \delta = 2\sqrt{mk} \]

\[ \text{If } \delta > \delta_{\text{critical}} \]

\[ \Rightarrow \]

\[ D > 0 \]

The nature of the general solution \( u(t) \) changes from periodic to non-periodic oscillations

from periodic oscillations to no oscillations
CASE 3: (Overdamping) \( D > 0 \) (two distinct real roots) In this case there are no oscillation.

The general solution of (2) has no more one zero:

\[
x(t) = e^{\lambda t}(C_1 e^{\sqrt{\lambda^2 - \omega^2} t} + C_2 e^{-\sqrt{\lambda^2 - \omega^2} t}).
\]
9. If $Q$ is the charge at time $t$ in an electrical closed circuit with inductance $L$, resistance $R$, and capacitance $C$, then by Kirchhoff’s Second Law (from Physics) the impressed voltage $E(t)$ is equal to the sum of the voltage drops in the rest of the circuit

$$E(t) = IR + \frac{Q}{C} + LI'(t).$$

By substitution $I = Q'$ we get

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Analogy between electrical and mechanical quantities:

- Charge $Q$  
  - Position $u$
- Inductance $L$  
  - mass $m$
- Resistance $R$  
  - Damping constant $\gamma$
- Inverse capacitance $1/C$  
  - Spring constant $k$
- Impressed voltage $E(t)$ (electromotive force)  
  - External force $F(t)$