Linear HOMOGENEOUS ODE of second order

8. Question: Can the function \( y = \sin(t^2) \) be a solution on the interval \((-1, 1)\) of a second order linear homogeneous equation with continuous coefficients?

9. Consider a linear homogeneous ODE

\[
y'' + p(t)y' + q(t)y = 0
\]

(2)

with coefficients \( p \) and \( q \) continuous in an interval \( I \).

10. Superposition Principle

- Sum \( y_1(t) + y_2(t) \) of any two solutions \( y_1(t) \) and \( y_2(t) \) of (2) is itself a solution.
- A scalar multiple \( Cy(t) \) of any solution \( y(t) \) of (2) is itself a solution.

COROLLARY 3. Any linear combination \( C_1y_1(t) + C_2y_2(t) \) of any two solutions \( y_1(t) \) and \( y_2(t) \) of (2) is itself a solution.

\[
y_1 = \cos t \\
y_2 = 5 \cos t \\
y = c_1y_1 + c_2y_2 = c_1 \cos t + 5c_2 \cos t = C \cos t
\]
11. Why Superposition Principle is important? Once two solutions of a linear homogeneous equation are known, a whole class of solutions is generated by linear combinations of these two.

\[ y'' + p(t)y' + q(t)y = 0 \quad (1) \]
\[ y(t_0) = y_0, \quad y'(t_0) = V_0 \quad (1*) \]

**IVP:**
\[ y(t) = C_1 y_1(t) + C_2 y_2(t) \quad (2) \]

Assume that \( y_1(t) \) and \( y_2(t) \) are particular solutions of \( (1) \). By Superposition Principle, \( y(t) \) is also solution of \( (1) \).

\( (2) \) is solution of IVP if and only if there exist \( C_1 \) and \( C_2 \) such that solution \( (2) \) satisfies the initial conditions \( (1*) \).

\[
\begin{array}{c|c}
\text{\( y(t_0) \)} & \text{\( C_1 \) \( y_1(t_0) \) \( + \) \( C_2 \) \( y_2(t_0) \)} \\
\text{\( y'(t_0) \)} & \text{\( C_1 \) \( y_1'(t_0) \) \( + \) \( C_2 \) \( y_2'(t_0) \)} \\
\end{array}
\]

By Cramer’s Rule

\[
W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0 \quad \text{then the above system has unique solution \( C_1, C_2 \)}
\]

\[
C_1 = \frac{y_0 - y_2(t_0)}{W(y_1, y_2)(t_0)} \quad \text{and} \quad C_2 = \frac{y_1(t_0) - y_1(t_0)}{W(y_1, y_2)(t_0)}
\]

12. Wronskian of the functions \( y_1(t) \) and \( y_2(t) \):

\[
W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}
\]
13. Suppose that \(y_1(t)\) and \(y_2(t)\) are two differentiable solutions of (2) in the interval \(I\) such that 
\(W(y_1, y_2)(t) \neq 0\) somewhere in \(I\), then every solution is a linear combination of \(y_1(t)\) and \(y_2(t)\).

In other words, the family of solutions \(y(t) = C_1y_1(t) + C_2y_2(t)\) with arbitrary coefficients \(C_1\) and \(C_2\) includes every solution of (2) if and only if there is a points \(t_0\) where \(W(y_1, y_2)\) is not zero. In this case the pair \((y_1(t), y_2(t))\) is called the fundamental set of solutions of (2).

**Remark 4.** Wronskian \(W(y_1, y_2)(t)\) (of any two solutions \(y_1(t)\) and \(y_2(t)\) of (2)) either is zero for all \(t\) or else is never zero.

For example, if \(y_1 = \cos t\), \(y_2 = 5 \cos t\) then

\[
W(y_1, y_2) = \begin{vmatrix}
\cos t & 5 \cos t \\
-\sin t & -5 \sin t
\end{vmatrix} = -5 \cos t \sin t - (-5 \cos t \sin t) = 0
\]

\[
\{y_1, y_2\} = \{\cos t, 5 \cos t\} \text{ is NOT fundamental set}
\]

And thus \(c_1 \cos t + c_2 \cdot 5 \cos t\) is not general solution of ODE of 2nd order.
14. Confirm that $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$. Then solve the IVP

$$y'' + y = 0, \quad y(\pi) = 0, \quad y'(\pi) = -5$$

$$y_1(x) = \sin x \implies y_1''(x) + \sin x = -\sin x + \sin x = 0$$

$$y_2(x) = \cos x \implies y_2'' + y_2 = (\cos x)'' + \cos x = -\cos x + \cos x = 0$$

Solve IVP. Determine whether $\{y_1, y_2\}$ is fundamental set ($\Rightarrow$ **Wrong** if $W(y_1, y_2) = 0$).

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$$

$$= -\left( \sin^2 x + \cos^2 x \right) = -1 \neq 0$$

$\{y_1, y_2\} = \{\sin x, \cos x\}$ is fundamental set,

$$y(t) = c_1 \sin x + c_2 \cos x \text{ is general solution}$$
Solve IVP: Find $c_1, c_2$ such that

$$y(t) = c_1 \sin t + c_2 \cos t$$

satisfies initial conditions

$$y(\pi) = 0, \quad y'(\pi) = -5$$

\[
y' = c_1 \cos t - c_2 \sin t
\]

\[
y(\pi) = c_1 \cdot 0 + c_2 \cdot (-1) = 0 \implies c_2 = 0
\]

\[
y'(\pi) = -c_1 + c_2 \cdot 0 = -5 \implies c_1 = 5
\]

Solution of IVP: 

$$y(t) = 5 \sin t$$
Appendix: Facts from Algebra

1. FACT 1: Cramer’s Rule for solving the system of equations

\[ \begin{align*}
    a_1x + b_1y &= c_1 \\
    a_2x + b_2y &= c_2
\end{align*} \]

The rule says is that if the determinant of the coefficient matrix is not zero, i.e.

\[ \left| \begin{array}{cc}
    a_1 & b_1 \\
    a_2 & b_2
\end{array} \right| \neq 0, \]

then the system has a unique solution \((x,y)\) given by

\[ x = \frac{\left| \begin{array}{cc}
    c_1 & b_1 \\
    c_2 & b_2
\end{array} \right|}{\left| \begin{array}{cc}
    a_1 & b_1 \\
    a_2 & b_2
\end{array} \right|}, \quad y = \frac{\left| \begin{array}{cc}
    a_1 & c_1 \\
    a_2 & c_2
\end{array} \right|}{\left| \begin{array}{cc}
    a_1 & b_1 \\
    a_2 & b_2
\end{array} \right|} \]
• FACT 2: If determinant of the coefficient matrix is zero then either there is no solution, or there are infinitely many solutions.

• FACT 3. The homogeneous system of linear equations

\[ a_1 x + b_1 y = 0 \]
\[ a_1 x + b_2 y = 0 \]

always has the “trivial” solution \((x, y) = (0, 0)\). By Cramer’s rule this is the only solution if the determinant of the coefficient matrix is not zero.

• FACT 4: If determinant of the coefficient matrix of homogeneous system of linear equations is zero then there are infinitely many nontrivial solutions \((x, y) \neq (0, 0)\).
2. Use Facts 1-4 to determine if each the following systems of linear equations has one solution, no solution, infinitely many solutions. Then find the solution(s) (if any).

(a) \[ \begin{align*}
2x + 3y &= 5 \\
x - y &= 4
\end{align*} \]

\[ \begin{vmatrix}
2 & 3 \\
1 & -1
\end{vmatrix} = -2 - 3 = -5 \neq 0 \]

\[ \Rightarrow \text{the system has a unique solution} \]

\[ x = \frac{5 - 12}{-5} = \frac{-7}{-5} = \frac{7}{5}, \quad y = \frac{1 - 4}{-5} = \frac{-3}{-5} = \frac{3}{5} \]

\[ (x, y) = \left( \frac{7}{5}, \frac{3}{5} \right) \]
(b) \[ 2x - 2y = 4 \]
    \[ x - y = 7 \]

(c) \[ 2x - 2y = 0 \]
    \[ 3x + 3y = 0 \]

(d) \[ 2x - 2y = 0 \]
    \[ 3x - 3y = 0 \]