Isolated Singularities of Nonlinear Elliptic Inequalities

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1. Statement of Problem

We discuss the growth near the origin of $C^2$ positive solutions $u(x)$ of
\[ af(u) \leq -\Delta u \leq f(u) \quad \text{in} \quad B^n - \{0\} \quad (1) \]
where $B^n = \{x \in \mathbb{R}^n: |x| < 1\}$, $a \in [0, 1)$ is a constant, and $f: (0, \infty) \to (0, \infty)$ is a continuous function.

It seems, at least for certain $f$, that studying (1) is more appropriate than studying $-\Delta u = f(u)$ in $B^n - \{0\}$.

**Question.** Does (1) have $C^2$ positive solutions which are arbitrarily large near the origin?

(i.e. for each continuous function $\varphi$: $(0, 1) \to (0, \infty)$ does there exist a $C^2$ positive solution $u(x)$ of (1) such that
\[ u(x) \neq \mathcal{O}(\varphi(|x|)) \quad \text{as} \quad |x| \to 0^+? \quad (2) \]

This is an important question because when the answer is no, one can usually show that all $C^2$ positive solutions of (1) are nearly radial near the origin.

Let $u$ be a $C^2$ positive solution of (1). Then $-\Delta u \geq 0$ in $B^n - \{0\}$. Thus when $n = 1$
\[ u(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to 0^+ , \]
and when $n \geq 2$
\[ u(x) \in L^1 \left( \frac{1}{2} B^n \right) . \]
Hence if $\varphi(r)$ is "large" (for example $\varphi(r) \geq \frac{1}{r^n}$) near $r = 0$ and $u(x)$ satisfies (2) then near the origin, $u(x)$ can be larger than $\varphi(|x|)$ only on a set of relatively small measure, that is
\[ \lim_{\varepsilon \to 0} \frac{|\{x \in B_\varepsilon(0) : u(x) > \varphi(|x|)\}|}{|B_\varepsilon(0)|} = 0 \]
When \( n = 2 \) and \( a = 0 \), the function \( f(t) = e^t \) is critical.

When \( n \geq 3 \) and \( a = 0 \), the function \( f(t) = t^{n/(n-2)} \) is critical.

When \( n \geq 3 \) and \( 0 < a < 1 \), the function \( f(t) = t^{(n+2)/(n-2)} \) is critical.

2. Results when \( n \geq 3 \) and \( a = 0 \)

First we state a result of Aviles for \(-\Delta u = u^{n/(n-2)}\).

**Theorem [A].** If \( u(x) \) is a \( C^2 \) positive solution of

\[
-\Delta u = u^{n/(n-2)} \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3
\]

then either \( u \) has a removable singularity at the origin or

\[
|x|^{n-2} \left( \log \frac{1}{|x|} \right)^{(n-2)/2} u(x) \to \ell \quad \text{as} \quad |x| \to 0^+
\]

where \( \ell = \ell(n) \) is a positive constant.

Consider now

\[
0 \leq -\Delta u \leq f(u) \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3. \quad (1)
\]

Critical growth rate for \( f \) is \( f(t) \sim t^{n/(n-2)} \) as \( t \to \infty \).

**Theorem 1.** The problem (1), where \( f: (0, \infty) \to (0, \infty) \) is any continuous function satisfying

\[
\lim_{t \to \infty} \frac{f(t)}{t^{n/(n-2)}} = \infty,
\]
has $C^2$ positive solutions which are arbitrarily large near the origin.

**Theorem 2.** Let $u(x)$ be a $C^2$ positive solution of (1) where $f : (0, \infty) \to (0, \infty)$ is a continuous function satisfying

$$f(t) = \mathcal{O}(t^{n/(n-2)}) \quad \text{as} \quad t \to \infty. \tag{2}$$

Then

$$u(x) = \mathcal{O}(|x|^{2-n}) \quad \text{as} \quad |x| \to 0^+ \tag{3}$$

and

$$0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad \text{for} \quad |x| \text{ small and positive} \tag{4}$$

where $\bar{u}(r)$ is the average of $u$ on the sphere $|x| = r$. Moreover, if $u^p$ is summable in some neighborhood of the origin for some $p > n/(n-2)$ then $u$ has a $C^1$ extension to the origin and $u(0) > 0$.

Theorem 2 is optimal in three ways. First, and most important, the growth condition (2) on $f(t)$ cannot be weakened because of Theorem 1.

Second, the conclusion (3) cannot be strengthened because $|x|^{2-n}$ is a $C^2$ positive solutions of (1).

And third, in the last sentence of Theorem 2, the condition on $p$ cannot be weakened to $p \geq n/(n-2)$ because for $(n-2)/n < \sigma \leq (n-2)/2$ the function

$$u_\sigma(x) := \left(\frac{n-2}{\sqrt{2}}\right)^{n-2} \frac{1}{|x|^{n-2} \left(\log \frac{1}{|x|}\right)^\sigma}$$
is a $C^2$ positive solution of

$$0 \leq -\Delta u \leq u^{n/(n-2)} \quad \text{in} \quad B^n - \{0\}$$

and $u_{\sigma}^{n/(n-2)}$ is summable in some neighborhood of the origin.

**Open Question.** Can (4) be strengthened to

$$\frac{u(x)}{\bar{u}(|x|)} \to 1 \quad \text{as} \quad |x| \to 0^+.$$

The following theorem extends Theorem 2 to the case $f$ depends on $|x|$.

**Theorem 3.** Let $u(x)$ be a $C^2$ positive solution of

$$0 \leq -\Delta u \leq |x|^{-\sigma} u^{n-\sigma} \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3$$

where $\sigma \in [0, n)$ is a constant. Then $u(x)$ satisfies (3) and (4).

3. Results when $n = 2$ and $a = 0$

Consider

$$0 \leq -\Delta u \leq f(u) \quad \text{in} \quad B^2 - \{0\}. \quad (1)$$

Critical growth rate for $f$ is $\log f(t) \sim t$ as $t \to \infty$.

**Theorem 1.** The problem (1), where $f: (0, \infty) \to (0, \infty)$ is any continuous function satisfying

$$\lim_{t \to \infty} \frac{\log f(t)}{t} = \infty,$$
has $C^2$ positive solutions which are arbitrarily large near the origin.

**Theorem 2.** Let $u(x)$ be a $C^2$ positive solution of (1) where $f: (0, \infty) \to (0, \infty)$ is a continuous function satisfying

$$\log f(t) = O(t) \quad \text{as} \quad t \to \infty. \quad (2)$$

Then

$$u(x) = O \left( \log \frac{1}{|x|} \right) \quad \text{as} \quad |x| \to 0^+. \quad (3)$$

Theorem 2 is optimal in two ways. First, and more important, the growth condition (2) on $f(t)$ cannot be weakened because of Theorem 1.

Second, the conclusion (3) cannot be strengthened because $\log \frac{1}{|x|}$ is a $C^2$ positive solutions of (1).

However we are not able to show that $u(x)$ in Theorem 2 satisfies

$$0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad \text{for} \ |x| \text{ small and positive}$$

where $\bar{u}(r)$ is the average of $u$ on the sphere $|x| = r$, but we make the

**Conjecture.** If $u$ is as in Theorem 2 then either

$$\frac{u(x)}{\log \frac{1}{|x|}} \to m \quad \text{as} \quad |x| \to 0^+ \quad (4)$$
for some positive constant \( m \) or \( u \) has a \( C^1 \) extension to the origin.

This conjecture is true if the condition on \( u \) is slightly strengthened. More precisely, if \( u \) is a \( C^2 \) positive solution in \( B^2 - \{0\} \) of either

\[
ae^u \leq -\Delta u \leq e^u \quad \text{or} \quad 0 \leq -\Delta u \leq f(u),
\]

where \( a \in (0,1) \) is a constant and \( f: (0, \infty) \to (0, \infty) \) is a continuous function satisfying

\[
\log f(t) = o(t) \quad \text{as} \quad t \to \infty
\]

then either \( u \) satisfies (4) for some positive constant \( m \) or \( u \) has a \( C^1 \) extension to the origin.

Two corollaries of Theorem 2 are

**Corollary 1.** Let \( v(x) \) be a \( C^2 \) solution of

\[
0 \leq -\Delta v \leq |x|^{-a} e^v
\]

\[
v(x) > -a \log \frac{1}{|x|} \quad \text{in} \quad B^2 - \{0\}
\]

where \( a \) is a positive constant. Then

\[
v(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as} \quad |x| \to 0^+.
\]

**Proof.** Let \( u(x) = v(x) + a \log \frac{1}{|x|} \). Then \( u(x) \) is a \( C^2 \) positive solution of

\[
0 \leq -\Delta u \leq e^u \quad \text{in} \quad B^2 - \{0\}.
\]

Thus Corollary 1 follows from Theorem 2.
Corollary 2. Let $u(x)$ be a $C^2$ solution of

$$
0 \leq -\Delta u \leq |x|^a e^u
$$

$$
u(x) > -a \log |x|
$$

in $\mathbb{R}^2 - \overline{B^2}$

where $a$ is a positive constant. Then

$$
u(x) = O(\log |x|) \quad \text{as} \quad |x| \to \infty.
$$

Proof. Apply the Kelvin transform $u(x) = v(y), \ x = y/|y|^2$

and then use Corollary 1.
Recall that $C^2$ solutions $u$ of
\[ 0 \leq -\Delta u \leq u^{n/(n-2)} \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3 \]
satisfy $u(x) = O(|x|^{2-n})$ as $|x| \to 0^+$. 

However the problem
\[ 0 \leq -\Delta u \leq u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2} \]
has arbitrarily large solutions near the origin.

Consider instead the more restricted problem
\[ au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad \frac{n}{n-2} < \lambda < \frac{n+2}{n-2} \]
where $0 < a < 1$.
Arbitrarily large solutions near the origin?
Answer depends on $a$.
Thus this is the correct problem to study for $\lambda$ as above.
4. Results when \( n \geq 3, 0 < a < 1 \) and \( f(t) = t^\lambda, \frac{n}{n-2} < \lambda < \frac{n+2}{n-2} \).

First we state a result of Gidas and Spruck for \(-\Delta u = u^\lambda\).

**Theorem [GS].** If \( u(x) \) is a \( C^2 \) positive solution of

\[-\Delta u = u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3\]

where the constant \( \lambda \) satisfies

\[
\frac{n}{n-2} < \lambda < \frac{n+2}{n-2} \tag{1}
\]

then either \( u \) has a removable singularity at the origin or

\[
|x|^{2/(\lambda-1)} u(x) \to \ell \quad \text{as} \quad |x| \to 0^+
\]

where \( \ell = \ell(n, \lambda) \) is a positive constant.

Consider now

\[
au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3 \tag{2}
\]

where

\[
\frac{n}{n-2} < \lambda < \frac{n+2}{n-2} \tag{1}
\]

**Theorem 1.** Suppose \( \lambda \) satisfies (1). Then there exists \( a = a(n, \lambda) \in (0, 1) \) such that (2) has \( C^2 \) positive solutions which are arbitrarily large near the origin.

**Theorem 2.** Suppose \( \lambda \) satisfies (1). Then there exists \( a = a(n, \lambda) \in (0, 1) \) such that if \( u \) is a \( C^2 \) positive solution of (2) then

\[
u(x) = \mathcal{O}(|x|^{-2/(\lambda-1)}) \quad \text{as} \quad |x| \to 0^+
\]
and

\[ 0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad \text{for } |x| \text{ small and positive} \]

where \( \bar{u}(r) \) is the average of \( u \) on the sphere \( |x| = r \).

Theorem 1 is not true when \( \lambda \leq n/(n - 2) \) by a previous result.

Let \( \lambda \) satisfy (1) and let

\[ I_1 = I_1(n, \lambda) = \{ a \in (0, 1) : \text{Theorem 1 is true} \} \]

\[ I_2 = I_2(n, \lambda) = \{ a \in (0, 1) : \text{Theorem 2 is true} \}. \]

Then \( I_1 \) and \( I_2 \) are nonempty disjoint subintervals of \( (0, 1) \).

**Open Question.** Does \( I_1 \cup I_2 = (0, 1) ? \) If not, what is the behavior of \( C^2 \) positive solutions of (2) when \( a \in (0, 1) - (I_1 \cup I_2) ? \)

**Local Behavior at \( \infty \).** Consider

\[ au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in } \mathbb{R}^n - B^n, \quad n \geq 3 \quad (3) \]

where (as before)

\[ \frac{n}{n - 2} < \lambda < \frac{n + 2}{n - 2} \quad (1) \]
Theorem 3. Suppose $\lambda$ satisfies (1). Then there exists $a = a(n, \lambda) \in (0, 1)$ such that (3) has $C^2$ positive solutions which are arbitrarily large near infinity.

Theorem 4. Suppose $\lambda$ satisfies (1). Then there exists $a = a(n, \lambda) \in (0, 1)$ such that if $u$ is a $C^2$ positive solution of (3) then

$$u(x) = O(|x|^{-2/(\lambda-1)}) \quad as \quad |x| \to \infty$$

and

$$0 < C_1 \leq \frac{u(x)}{\bar{u}(|x|)} \leq C_2 < \infty \quad for \quad |x| \text{ large}$$

where $\bar{u}(r)$ is the average of $u$ on the sphere $|x| = r$.

Theorem 3 is not true when $\lambda \leq n/(n-2)$. In fact Serrin and Zou prove that if $0 \leq \lambda \leq n/(n-2)$ then there does not exist a $C^2$ positive solution of $-\Delta u \geq u^\lambda$ in $\mathbb{R}^n - B^n$ and this result can be easily extended to include all negative values of $\lambda$.

Global Existence. First we state a result of Gidas and Spruck for $-\Delta u = u^\lambda$.

Theorem [GS]. There does not exist a $C^2$ positive solution of

$$-\Delta u = u^\lambda \quad in \quad \mathbb{R}^n, \quad n \geq 3$$

when the constant $\lambda$ satisfies.

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \quad (1)$$

Consider now

$$au^\lambda \leq -\Delta u \leq u^\lambda \quad in \quad \mathbb{R}^n, \quad n \geq 3 \quad (4)$$

where $\lambda$ satisfies (1).
Theorem 5. Suppose $\lambda$ satisfies (1). Then there exists $a = a(n, \lambda) \in (0, 1)$ such that there exists a $C^2$ positive solution $u(x)$ of (4).

Theorem 6. Suppose $\lambda$ satisfies (1). Then there exists $a = a(n, \lambda) \in (0, 1)$ such that there does not exist a $C^2$ positive solution $u(x)$ of (4).

When $\lambda$ satisfies (1), these six theorems show that changing $a$ from one value in the open interval $(0, 1)$ to another value in $(0, 1)$ can dramatically affect the local behavior and global existence of $C^2$ positive solutions of $au^\lambda \leq -\Delta u \leq u^\lambda$.

Consider the

Question. For what values of $\lambda \in \mathbb{R}$ and $a \in (0, 1]$ does there exist a $C^2$ positive solution of

$$au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3? \quad (4)$$

To answer this question we consider three mutually exclusive possibilities for the value of $\lambda$.

Case I. Suppose $-\infty < \lambda \leq n/(n-2)$. Then for each $a \in (0, 1]$, (4) does not have a $C^2$ positive solution by Serrin and Zou’s result mentioned above.

Case II. Suppose

$$\frac{n}{n-2} < \lambda < \frac{n+2}{n-2}. \quad (1)$$

Then our Theorems 5 and 6 hold.

Case III. Suppose $(n+2)/(n-2) \leq \lambda < \infty$. Then for each $a \in (0, 1]$, (4) has a $C^2$ positive solution because Fowler shows that (4) with $a = 1$ has a $C^2$ positive radial solution.
5. Results when $n \geq 3$, $0 < a < 1$ and $f(t) = t^{\frac{n+2}{n-2}}$. 

Caffarelli, Gidas, and Spruck proved

**Theorem [CGS].** If $u(x)$ is a $C^2$ positive solution of

$$-\Delta u = u^{(n+2)/(n-2)} \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3$$

then either $u$ has a removable singularity at the origin or

$0 < C_1 \leq |x|^{(n-2)/2}u(x) \leq C_2 < \infty$  \quad \text{for} \quad |x| \text{ small and positive.}

Consider now

$$au^{\frac{n+2}{n-2}} \leq -\Delta u \leq u^{\frac{n+2}{n-2}} \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3. \quad (1)$$

**Theorem.** The problem (1) has $C^2$ positive solutions which are arbitrarily large near the origin provided $0 < a < 2^{-4/(n-2)}$.

**Conjecture.** If $2^{-4/(n-2)} < a < 1$ and $u$ is a $C^2$ positive solution of (1) then

$$u(x) = O\left(\frac{1}{|x|^{(n-2)/2}}\right) \quad \text{as} \quad |x| \to 0^+.$$ 

6. Results when $n \geq 3$ and $f(t) = t^\lambda$, $\lambda > \frac{n+2}{n-2}$.

**Theorem.** Suppose $\lambda > \frac{n+2}{n-2}$. Then there exists

$a = a(n, \lambda) \in (0, 1)$ such that

$$au^\lambda \leq -\Delta u \leq u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad n \geq 3$$

has $C^2$ positive solutions which are arbitrarily large near the origin.

**Conjecture.** The problem

$$-\Delta u = u^\lambda \quad \text{in} \quad B^n - \{0\}, \quad \text{where} \quad n \geq 3 \quad \text{and} \quad \lambda > \frac{n+2}{n-2},$$

has $C^2$ positive solutions which are arbitrarily large near the origin.
Theorem. The problem

\[ 0 \leq -\Delta u \leq f(u) \quad \text{in} \quad B^2 - \{0\}, \]

where \( f: (0, \infty) \to (0, \infty) \) is any continuous function satisfying

\[ \lim_{t \to \infty} \frac{\log f(t)}{t} = \infty, \quad (1) \]

has \( C^2 \) positive solutions which are arbitrarily large near the origin.

Proof. By scaling \( x \), we can replace \( B^2 \) with \( D = \{ x \in \mathbb{R}^2 : |x| < 1/2 \} \). Assume for simplicity that \( f \) is increasing. Let \( \varphi: (0, 1) \to (0, \infty) \) be a continuous function. For \( k = 1, 2, \ldots \), let \( x^k = (1/2^{2k}, 0) \in \mathbb{R}^2 \),

\[ r_k \in (0, 1/2^{2k+1}), \quad (2) \]

and \( h_k^2 = r_k^2 / 2 \).

Then the balls \( B_{r_k}(x^k) \) are pairwise disjoint. Let

\[ g_k(x) = \begin{cases} \frac{L_k}{\pi h_k^2}, & \text{for } |x - x^k| < h_k \\ 0, & \text{elsewhere in } D \end{cases} \]

where

\[ L_k \in (0, 1) \quad \text{and} \quad \sum_{k=1}^{\infty} L_k < \infty. \quad (3) \]
Let
\[ \gamma_k(x) = \begin{cases} \frac{r_k - |x - x^k|}{r_k - h_k} g_k(x^k), & \text{for } h_k \leq |x - x^k| \leq r_k \\ 0, & \text{elsewhere in } D. \end{cases} \]

Define \( g : D \to [0, \infty) \) by
\[ g(x) = \sum_{k=1}^{\infty} (g_k(x) + \gamma_k(x)). \]

Then
\[ \int_D g(x) dx = \sum_{k=1}^{\infty} \int_{B_{r_k}(x^k)} (g_k(x) + \gamma_k(x)) dx \leq \sum_{k=1}^{\infty} |B_{r_k}(x^k)| g_k(x^k) = \sum_{k=1}^{\infty} 2L_k < \infty. \]

Thus defining \( u : D - \{0\} \to (0, \infty) \) by
\[ u(x) = (Ng)(x) := \frac{1}{2\pi} \int_D \left( \log \frac{1}{|x - y|} \right) g(y) dy \]
we have \( u \in C^2(D - \{0\}) \) and
\[ -\Delta u = g \geq 0 \quad \text{in } D - \{0\}. \]

To complete the proof we show for each sequence \( \{L_k\}_{k=1}^{\infty} \) satisfying (3) there exists a sequence \( \{r_k\}_{k=1}^{\infty} \) satisfying (2) such that
\[ -\Delta u \leq f(u) \quad \text{in } D - \{0\} \quad (4) \]
and
\[ \frac{u(x^k)}{\varphi(|x^k|)} \to \infty \quad \text{as} \quad k \to \infty. \quad (5) \]

Since
\[ (N g_k)(x) = \begin{cases} \frac{L_k}{4\pi} \left( 1 - \frac{|x-x^k|^2}{h_k^2} \right) + \frac{L_k}{2\pi} \log \frac{1}{h_k}, & |x-x^k| \leq h_k \\ \frac{L_k}{2\pi} \log \frac{1}{|x-x^k|}, & |x-x^k| \geq h_k \end{cases} \]
we have
\[ u(x^k) \geq (N g_k)(x^k) \geq \frac{L_k}{2\pi} \log \frac{1}{r_k} =: K_k \]
and hence (5) holds provided
\[ \frac{K_k}{\varphi(|x^k|)} \to \infty \quad \text{as} \quad k \to \infty. \quad (6) \]

(4) clearly holds for \( x \in D - \{0\} - \bigcup_{k=1}^\infty B_{r_k}(x^k) \). Also, for \( x \in B_{r_k}(x^k) \) we have
\[ f(u(x)) \geq f((N g_k)(x)) \geq f \left( \frac{L_k}{2\pi} \log \frac{1}{r_k} \right) = f(K_k) \]
and
\[ -\Delta u(x) = g(x) \leq g_k(x^k) = \frac{L_k}{\pi h_k^2} = \frac{2L_k}{\pi r_k^2} \leq \frac{1}{r_k^2}. \]
Thus (4) holds in \( B_{r_k}(x^k) \) provided
\[ \log f(K_k) > \log \frac{1}{r_k^2} = 2 \log \frac{1}{r_k} = \frac{4\pi}{L_k} K_k. \quad (7) \]

Also, (2) requires that
\[ \exp \left( -\frac{2\pi K_k}{L_k} \right) = r_k < \frac{1}{2^{2k+1}}. \quad (8) \]
By (1) we can satisfy (6), (7), and (8) by choosing \( K_k \) sufficiently large. Then \( r_k \) is given in terms of \( K_k \) by (8).
**Theorem.** Let \( u(x) \) be a \( C^2 \) positive solution of

\[
0 \leq -\Delta u \leq e^u \quad \text{in} \quad B^2 - \{0\}.
\]

Then

\[
u(x) = O\left(\log \frac{1}{|x|}\right) \quad \text{as} \quad |x| \to 0^+.
\]

**Proof.** Let \( \Omega = B_{1/2}(0) \). Since \( u \) is positive and superharmonic in \( B_1(0) - \{0\} \) we have \( u, -\Delta u \in L^1(\Omega) \) and there exist a nonnegative constant \( m \) and a continuous function \( h: \overline{\Omega} \to \mathbb{R} \) which is harmonic in \( \Omega \) such that

\[
u(x) = m \log \frac{1}{|x|} + N(x) + h(x) \quad \text{for} \quad x \in \overline{\Omega} - \{0\}
\]

where

\[
N(x) = \frac{1}{2\pi} \int_{\Omega} \left(\log \frac{1}{|x - y|}\right) (-\Delta u(y)) dy.
\]

Suppose for contradiction there exists a sequence \( \{x_k\}_{k=1}^{\infty} \subset \frac{1}{2} \Omega - \{0\} \) such that \( |x_k| \to 0 \) as \( k \to \infty \) and

\[
limit_{k \to \infty} \frac{u(x_k)}{\log \frac{1}{|x_k|}} = \infty.
\]

Since, for \( |x - x_k| < |x_k|/4, \)

\[
\int_{|y-x_k|>|x_k|/2} \left(\log \frac{1}{|x - y|}\right) (-\Delta u(y)) dy \leq \left(\log \frac{4}{|x_k|}\right) \int_{\Omega} -\Delta u(y) dy
\]
it follows from (2) and (3) that

\[ u(x) \leq C \log \frac{1}{|x_k|} + \frac{1}{2\pi} \int_{|y-x_k|<|x_k|/2} \left( \log \frac{1}{|x-y|} \right) (-\Delta u(y)) dy \]

(5)

for \(|x-x_k| < |x_k|/4\) where \(C\) is a positive constant which does not depend on \(k\) or \(x\).

Substituting \(x = x_k\) in (5) and using (4) we obtain

\[ \frac{1}{\log \frac{1}{|x_k|}} \int_{|y-x_k|<|x_k|/2} \left( \log \frac{1}{|x_k-y|} \right) (-\Delta u(y)) dy \to \infty \quad \text{as} \quad k \to \infty. \]

(6)

Also, since \(-\Delta u \in L^1(\Omega)\) we have

\[ \int_{|y-x_k|<|x_k|/2} -\Delta u(y) dy \to 0 \quad \text{as} \quad k \to \infty. \]

(7)

For each positive integer \(k\), define a continuous function \(f_k: B_2 \to [0, \infty)\) (where \(B_r = B_r(0)\)) by

\[ f_k(\xi) = -r_k^2 \Delta u(x) \]

where \(x = x_k + r_k\xi\) and \(r_k = |x_k|/4\).

Making the change of variables \(y = x_k + r_k\zeta\) in (7), (6), and (5), and using (1) we obtain

\[ \int_{B_2} f_k(\zeta) d\zeta \to 0 \quad \text{as} \quad k \to \infty \]

(8)
\[
\frac{1}{M_k} \int_{B_2} \left( \log \frac{4}{|\zeta|} \right) f_k(\zeta) d\zeta \to \infty \quad \text{as} \quad k \to \infty \quad (9)
\]

\[
f_k(\xi) \leq \exp \left( M_k + \frac{1}{2\pi} \int_{B_2} \left( \log \frac{4}{|\xi - \zeta|} \right) f_k(\zeta) d\zeta \right) \quad \text{for} \quad \xi \in B_1
\]  

(10)

where \( M_k = C \log \frac{1}{|x_k|} \) and \( C \) is a positive constant which does not depend on \( k \) or \( \xi \).

Let \( \Omega_k = \{ \xi \in B_1: u_k(\xi) > M_k \} \) where

\[
u_k(\xi) = \frac{1}{2\pi} \int_{B_2} \left( \log \frac{4}{|\xi - \zeta|} \right) f_k(\zeta) d\zeta.
\]

Then letting \( p_k = \pi/\int_{B_2} f_k(\zeta) d\zeta \) it follows from (10) that

\[
\int_{\Omega_k} f_k(\xi)^{p_k} d\xi \leq \int_{B_2} e^{2p_k u_k(\xi)} d\xi
\]

\[
= \int_{B_2} \exp \left( \int_{B_2} \left( \log \frac{4}{|\xi - \zeta|} \right) \frac{f_k(\zeta)}{\int_{B_2} f_k} d\zeta \right) d\xi
\]

\[
\leq \int_{B_2} \left( \int_{B_2} \frac{4}{|\xi - \zeta|} \frac{f_k(\zeta)}{\int_{B_2} f_k} d\zeta \right) d\xi
\]

by Jensen’s inequality (Brezis and Merle)

\[
= \int_{B_2} \frac{f_k(\zeta)}{\int_{B_2} f_k} \left( \int_{B_2} \frac{4}{|\xi - \zeta|} d\xi \right) d\zeta \leq 16\pi.
\]
Thus by Hölder’s inequality

\[
\int_{\Omega_k} \left( \log \frac{4}{|\zeta|} \right) f_k(\zeta) d\zeta \leq \left\| \log \frac{4}{|\zeta|} \right\|_{L^{p'(B_1)}} \left\| f_k \right\|_{L^p(\Omega_k)}
\]

\[
\leq |B_1| \frac{1}{p_k} - \frac{1}{2} \left\| \log \frac{4}{|\zeta|} \right\|_{L^2(B_1)} (16\pi)^{1/p_k}
\]

\[
\leq (16\pi)^{1/2} \left\| \log \frac{4}{|\zeta|} \right\|_{L^2(B_1)} < \infty.
\]

Hence, defining \( g_k : B_1 \rightarrow [0, \infty) \) by

\[
g_k(\xi) = \begin{cases} 
  f_k(\xi), & \text{for } \xi \in B_1 - \Omega_k \\
  0, & \text{for } \xi \in \Omega_k
\end{cases}
\]

it follows from (9) and (8) that

\[
\frac{1}{M_k} \int_{B_1} \left( \log \frac{4}{|\zeta|} \right) g_k(\zeta) d\zeta
\]

\[
= \frac{1}{M_k} \int_{B_1} \left( \log \frac{4}{|\zeta|} \right) f_k(\zeta) d\zeta
\]

\[
- \frac{1}{M_k} \int_{\Omega_k} \left( \log \frac{4}{|\zeta|} \right) f_k(\zeta) d\zeta \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty. \quad (11)
\]

By (8) we have

\[
\int_{B_1} g_k(\zeta) d\zeta \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad (12)
\]
and by (10) we have

\[ g_k(\xi) \leq e^{2M_k} \quad \text{in} \quad B_1. \quad (13) \]

For fixed \( k \) think of \( g_k(\zeta) \) as the density of a distribution of mass in \( B_1 \) satisfying (11), (12), and (13). By moving small pieces of this mass nearer to the origin in such a way that the new density (which we again denote by \( g_k(\zeta) \)) does not violate (13), we will not change the total mass \( \int_{B_1} g_k(\zeta) d\zeta \) but \( \int_{B_1} \left( \log \frac{4}{|\zeta|} \right) g_k(\zeta) d\zeta \) will increase. Thus for some \( \rho_k \in (0, 1) \) the functions

\[ g_k(\zeta) = \begin{cases} 
    e^{2M_k}, \quad &\text{for } |\zeta| < \rho_k \\
    0, \quad &\text{for } \rho_k < |\zeta| < 1 
\end{cases} \]

satisfy (11), (12), and (13), which, as elementary and explicit calculations show, is impossible because \( M_k \to \infty \) as \( k \to \infty \). This contradiction proves the Theorem.