1. (15) Define each of the following.

(a) \( \lim_{x \to 3} f(x) = 4 \)

For any \( \epsilon > 0 \), there is a \( \delta > 0 \), such that if

\[ 0 < |x - 3| < \delta, \text{ then } |f(x) - 4| < \epsilon. \]

(b) \( f \) is continuous at the point \( x = -5 \).

\[ \lim_{x \to -5} f(x) = f(-5). \]

(c) The derivative of \( f \) at the point \( x = 2 \).

\[ f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}. \]

2. (15) Let \( \vec{x} = \langle 5, -3 \rangle \) and \( \vec{y} = \langle 2, 3 \rangle \).

(a) Compute the length of the vector \( \vec{x} \).

\[ \| \vec{x} \| = \sqrt{5^2 + (-3)^2} = \sqrt{34}. \]

(b) What is the angle between the vectors \( \vec{x} \) and \( \vec{y} \)?

\[ \cos \theta = \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \| \| \vec{y} \|} = \frac{10 + (-9)}{\sqrt{34} \sqrt{13}} = \frac{1}{\sqrt{442}}. \]

\[ \theta = \arccos \left( \frac{1}{\sqrt{442}} \right). \]

(c) Find vectors \( \vec{p} \) and \( \vec{q} \) such that \( \vec{p} \) is parallel to \( \vec{x} \), \( \vec{q} \) is perpendicular to \( \vec{x} \) and

\[ \vec{y} = \vec{p} + \vec{q} \]

\[ \vec{p} = \text{Proj}_{\vec{x}} \vec{y} = \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \|^2} \vec{x} = \frac{1}{34} \langle 5, -3 \rangle \]

\[ \vec{q} = \vec{y} - \vec{p} = \langle 2, 3 \rangle - \frac{1}{34} \langle 5, -3 \rangle = \frac{21}{34} \langle 3, 5 \rangle. \]

3. (15) Suppose that \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \). Prove that

\[ \lim_{x \to a} [f(x) + g(x)] = F + G \]

Let \( \delta_1 \) be such that if \( 0 < |x - a| < \delta_1 \), then \( |f(x) - F| < \frac{\epsilon}{2} \). Let \( \delta_2 \) be such that if \( 0 < |x - a| < \delta_2 \), then \( |g(x) - G| < \frac{\epsilon}{2} \). Set \( \delta = \min \{ \delta_1, \delta_2 \} \). Then if \( 0 < |x - a| < \delta \), we have

\[ |(f(x) + g(x)) - (F + G)| \leq |f(x) - F| + |g(x) - G| \]

\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
4. (15) Evaluate the following limits

(a) \( \lim_{x \to 0^-} \frac{|x|}{x} \)

Since \( x \) is approaching 0 from below we know that \( x < 0 \). Thus,
\[ \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1. \]

(b) \( \lim_{x \to \infty} \frac{x^2 - 3x + 1}{3x^2 - 5x} \)
\[
\lim_{x \to \infty} \frac{x^2 - 3x + 1}{3x^2 - 5x} = \lim_{x \to \infty} \frac{1 - 3/x + 1/x^2}{3 - 5/x} = \lim_{x \to \infty} \frac{(1 - 3/x + 1/x^2)}{(3 - 5/x)} = \frac{1}{3}.
\]

(c) \( \lim_{x \to 2} \frac{x^3 - x^2 - x - 2}{x - 2} \)
\[
\lim_{x \to 2} \frac{x^3 - x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + x + 1)}{x - 2} = \lim_{x \to 2} (x^2 + 1) = 7.
\]

5. (20) Let \( f(x) = -x^2 + 1 \).

(a) Using the definition of the derivative, calculate \( f'(3) \).
\[
f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{(-(h + 3)^2 + 1) - (-8)}{h} = \lim_{h \to 0} \frac{-6h - h^2 - 8}{h} = \lim_{h \to 0} -6h - h^2 = -6.
\]

(b) Find the equation for the tangent line to the graph of \( f \) at the point (3, -8).
\[
\frac{y - (-8)}{x - 3} = -6 \quad \Rightarrow \quad y = -6(x - 3) - 8 = 10 - 6x.
\]
(a) State the Intermediate Value Theorem for continuous functions.

Let \( f \) be continuous on the closed interval \([a, b]\). Suppose \( y_0 \) lies strictly between the numbers \( f(a) \) and \( f(b) \). Then there is a point \( c \) in the open interval \((a, b)\) such that \( f(c) = y_0 \).

(b) Use the method of bisection to estimate the value of \( \sqrt{7} \) to within \( \frac{1}{8} \).

Set \( f(x) = x^2 - 7 \). Then we are looking for a number \( c > 0 \) such that \( f(c) = c^2 - 7 = 0 \). What we want to find are numbers \( x_1 \) and \( x_2 \) such that there is a sign change between \( f(x_1) \) and \( f(x_2) \) and such that \( |x_1 - x_2| \leq 1/4 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( f(2) = -3 )</td>
</tr>
<tr>
<td>3</td>
<td>( f(3) = 2 )</td>
</tr>
<tr>
<td>5/2</td>
<td>( f(5/2) = -3/4 )</td>
</tr>
<tr>
<td>11/4</td>
<td>( f(11/4) = 9/16 )</td>
</tr>
</tbody>
</table>

From the above table we see that \( f(5/2) < 0 \) and \( f(11/4) > 0 \). Thus, our solution \( c \) lies between \( 5/2 \) and \( 11/4 \). As a rough guess to \( c \) we pick the midpoint of the two numbers \( 5/2 \) and \( 11/4 \), which we call \( x_0 \). Thus,

\[
x_0 = \frac{1}{2} \left( \frac{5}{2} + \frac{11}{4} \right) = \frac{21}{8}.
\]

Moreover we must have

\[
|c - x_0| \leq \frac{1}{2} \left( \frac{11}{4} - \frac{5}{2} \right) = \frac{1}{8}.
\]

The decimal expansion of \( x_0 \) is 2.625, and a decimal approximation to \( \sqrt{7} \) is 2.6458. So we’re not too far off with \( x_0 \). The actual error is about

\[
2.6458 - 2.625 = 0.0208.
\]

Much better than our 1/8 estimate.