1. (35) Limits
   a. Define what \( \lim_{x \to 5} f(x) = 3 \) means.

      For every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if
      \[ 0 < |x - 5| < \delta \text{ then } |f(x) - 3| < \epsilon. \]

   b. Using the definition of limit show that \( \lim_{x \to 1} \frac{2x - 1}{3} = \frac{1}{3} \).

      Let \( \epsilon > 0 \). Set \( \delta = 3\epsilon/2 \), then if \( 0 < |x - 1| < \delta = 3\epsilon/2 \), we have
      \[
      \left| \frac{2x - 1}{3} - \frac{1}{3} \right| = \left| \frac{2x - 2}{3} \right| = \frac{2}{3} |x - 1| < \frac{2}{3} \frac{3\epsilon}{3} = \epsilon.
      \]

   c. Calculate the following limits:

      i. \( \lim_{x \to \infty} \frac{n(2n + 1)(3n + 2)}{n^3} = 6 \)

      ii. \( \lim_{x \to 0} \frac{1}{x + 1} = -1 \)

      iii. After writing \( \left(1 - \frac{2}{x}\right)^{x/3} = e^{x \ln(1 - 2/x)^{1/3}} \), L’Hospital’s rule is used to evaluate the limit of the exponent.

         \[
         \lim_{x \to \infty} x \ln(1 - 2/x) \frac{1}{3} = \lim_{x \to \infty} \frac{\ln(1 - 2/x)}{3/x}
         \]

         \[
         = \lim_{x \to \infty} \frac{1 - 2/x}{-3/x^2} \frac{2/x^2}{(-2/x^2)}
         \]

         \[
         = -\frac{2}{3}.
         \]

         Thus, \( \lim_{x \to \infty} \left(1 - \frac{2}{x}\right)^{x/3} = e^{-2/3} \).

      iv. \( \lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x} = \lim_{x \to 0} \frac{\sin x}{6x} = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \)

      v. \( \lim_{x \to 0} \frac{\cos^{-1} x - \pi/2}{x} = \lim_{x \to 0} \frac{-1}{\sqrt{1 - x^2}} = -1 \).
2. (20) Give the domain, range and derivative for each of the following functions:
   
   \[ f \quad e^x \quad \ln x \quad \sin^{-1}x \quad \frac{1}{\sqrt{1+x}} \]

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<thead>
<tr>
<th>f</th>
<th>domain</th>
<th>range</th>
<th>derivative</th>
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<tbody>
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<td>(-\infty &lt; x &lt; \infty)</td>
<td>(0 &lt; x &lt; \infty)</td>
<td>(-1 \leq x \leq 1)</td>
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<td>(0 &lt; x &lt; \infty)</td>
<td>(-\infty &lt; x &lt; \infty)</td>
<td>(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})</td>
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<td>(\frac{1}{\sqrt{1-x^2}})</td>
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3. (30)
   
   a. Define the derivative of a function \(f\) at the point \(x = 3\).
   
   \[ f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \]
   
   b. Using the definition of the derivative determine \(f'(2)\) for \(f(x) = x^2 + 1\).
   
   \[ f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)^2 + 1] - 5}{h} \]
   
   \[ = \lim_{h \to 0} \frac{[5 + 4h + h^2] - 5}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} \]
   
   \[ = \lim_{h \to 0} 4 + h = 4 \]
   
   c. Calculate the derivatives of the following functions:
   
   i. \(\frac{2x+1}{\sin x}\)
   
   ii. \(\sin(x^2)\)
   
   iii. \((e^{2x} + \ln x)(4x + 1)^{10}\)
   
   iv. \(\tan^{-1}(3x)\)

   \[ \begin{align*}
   \text{i.} & \quad \frac{d}{dx} \frac{2x+1}{\sin x} = \frac{2 \sin x - (2x+1) \cos x}{\sin^2 x} \\
   \text{ii.} & \quad \frac{d}{dx} \sin(x^2) = 2x \cos(x^2) \\
   \text{iii.} & \quad \frac{d}{dx} (e^{2x} + \ln x)(4x + 1)^{10} = (2e^{2x} + 1/x)(4x + 1)^{10} + 40(e^{2x} + \ln x)(4x + 1)^9 \\
   \text{iv.} & \quad \frac{d}{dx} \tan^{-1}(3x) = \frac{3}{1 + (3x)^2}. 
   \end{align*} \]

4. (15) Elizabeth wants to make a tin can using as little material as possible. If the volume of the can is 64 cubic inches, what should the radius and height of the can equal?

   Let \(r\) and \(h\) denote the radius and height of the tin can respectively. The amount of material used is essentially the surface area of the can. So we want to find those dimensions that minimize the surface area.

   \[ V = \pi r^2 h = 64 \] Thus, \(h = \frac{64}{\pi r^2}\)

   \[ A = 2\pi r^2 + 2\pi rh \]

   \[ = 2\pi r^2 + 2\pi r \left( \frac{64}{\pi r^2} \right) = 2\pi r^2 + \left( \frac{128}{r} \right) \]

   Note that \(A\) becomes infinite as \(r\) approaches 0 or \(\infty\). Taking the derivative of \(A\) and setting it equal to zero we get
\[ A'(r) = 4\pi r - \frac{128}{r^2} = 0 \]

\[ r^3 = \frac{32}{\pi} \]

\[ r = \left(\frac{32}{\pi}\right)^{1/3} \]

\[ h = \frac{64}{\pi r^2} = \frac{64}{\pi (32/\pi)^{2/3}} \]

\[ = \frac{4}{\pi^{1/3}} 2^{2/3} \]

5. (20) Let \( f(x) = \begin{cases} x^2 \ln x, & x > 0 \\ 0, & x = 0 \end{cases} \).

a. Is \( f \) continuous from the right at \( x = 0 \)?

\( f \) is continuous from the right. To see this it suffices to show that \( \lim_{x \to 0^+} f(x) = f(0) = 0 \).

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x^2} \\
= \lim_{x \to 0^+} \left( \frac{1/x}{-2/x^3} \right) = \lim_{x \to 0^+} \frac{x^2}{2} \\
= 0.
\]

b. Where is \( f \) increasing or decreasing?

\[
f'(x) = \begin{cases} 2x \ln x + x, & x > 0 \\ 0, & x = 0 \end{cases}.
\]

Thus for \( 0 < x < e^{-1/2} \), \( f' < 0 \) and for \( x > e^{-1/2} \), \( f' > 0 \). So \( f \) is decreasing on \([0, e^{-1/2}]\) and increasing on \([e^{-1/2}, \infty)\).

c. Where is \( f \) concave up or concave down?

\[
f''(x) = \begin{cases} 2 \ln x + 3, & x > 0 \\ \text{DNE}, & x = 0 \end{cases}.
\]

So for \( 0 < x < e^{-3/2} \) the second derivative is negative and for \( x > e^{-3/2} \) the second derivative is positive. Thus, the graph of \( f \) is concave down on \((0, e^{-3/2})\) and concave up on \((e^{-3/2}, \infty)\).

d. Graph \( f \), be sure to indicate any asymptotic behavior.

Note that \( \lim_{x \to \infty} f(x) = \infty \).
6. (15) The point \( \left( \frac{\pi}{2}, 1 \right) \) satisfies the equation \( x + \sin x + x \ln y + y^2 = 2 + \frac{\pi}{2} \).

a. Find \( y' \left( \frac{\pi}{2} \right) \).

Differentiating the given equation with respect to \( x \) we have

\[
1 + \cos x + \ln y + \frac{x}{y} y' + 2yy' = 0 .
\]

Solving for \( y' \) we get

\[
y' \left| \left( \frac{\pi}{2}, 1 \right) \right. = - \frac{1 + 0 + 0}{2 + \pi/2} \]
\[
= - \frac{2}{4 + \pi} \approx -0.280 .
\]

b. Use the tangent line to the curve given by the above equation to approximate to \( y \left( \frac{\pi}{2} + \frac{1}{10} \right) \).

\[
y \left( \frac{\pi}{2} + \frac{1}{10} \right) \approx y \left( \frac{\pi}{2} \right) + y' \left( \frac{\pi}{2} \right) \left( \frac{1}{10} \right)
\]
\[
= 1 - \left( \frac{2}{4 + \pi} \right) \frac{1}{10}
\]
\[
= 1 - \left( \frac{1}{20 + 5\pi} \right)
\]
\[
\approx 0.972 .
\]

7. (15) Let \( f(x) = x^2 \). Let \( P \) denote the partition of \([-1, 3]\) that divides \([-1, 3]\) into \( n \) equal length subintervals. Thus, \( P = \{x_0, x_1, \ldots, x_n\} \) and the distance between the adjacent points in \( P \) is a constant.

a. Express \( x_i \) in terms of the endpoints of \([-1, 3]\) and \( i \).

\[
x_i = -1 + \frac{4i}{n}, \quad i = 0, 1, \ldots, n.
\]

b. Using the definition of the definite integral calculate \( \int_{-1}^{3} x^2 \, dx \).

\[
\int_{-1}^{3} x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( -1 + \frac{4i}{n} \right)^2 \frac{4}{n}
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 - \frac{8i}{n} + \frac{16i^2}{n^2} \right) \frac{4}{n}
\]
\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{4}{n} - \lim_{n \to \infty} \sum_{i=1}^{n} \frac{32i}{n^2} + \lim_{n \to \infty} \sum_{i=1}^{n} \frac{64i^2}{n^3}
\]
\[
= \lim_{n \to \infty} \frac{4}{n}n - \lim_{n \to \infty} \frac{32}{n^2} \frac{n(n+1)}{2} + \lim_{n \to \infty} \frac{64}{n^3} \frac{n(n+1)(2n+1)}{6}
\]
\[
= 4 - 16 + \frac{64}{3}
\]
\[
= \frac{28}{3} .
\]