Math 304 Answers to Selected Problems

1 Section 4.1

4. Let $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear operator. If

$$L \left( (1, 2)^T \right) = (-2, 3)^T \quad \text{and} \quad L \left( (1, -1)^T \right) = (5, 2)^T$$

determine the value of $L \left( (7, 5)^T \right)$.

**Answer:** First, we write $\left( \begin{array}{c} 7 \\ 5 \end{array} \right)$ as a linear combination of $\left( \begin{array}{c} 1 \\ 2 \end{array} \right)$ and $\left( \begin{array}{c} 1 \\ -1 \end{array} \right)$:

$$c_1 \left( \begin{array}{c} 1 \\ 2 \end{array} \right) + c_2 \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = \left( \begin{array}{c} 7 \\ 5 \end{array} \right)$$

This gives us the equations

$$c_1 + c_2 = 7$$
$$2c_1 - c_2 = 5$$

Solving for $c_1$ and $c_2$, we get $c_1 = 4$ and $c_2 = 3$. Thus,

$$L \left( \begin{array}{c} 7 \\ 5 \end{array} \right) = L \left( 4 \left( \begin{array}{c} 1 \\ 2 \end{array} \right) + 3 \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right)$$

$$= 4L \left( \begin{array}{c} 1 \\ 2 \end{array} \right) + 3L \left( \begin{array}{c} 1 \\ -1 \end{array} \right)$$

$$= 4 \left( \begin{array}{c} 1 \\ -2 \\ 3 \end{array} \right) + 3 \left( \begin{array}{c} 5 \\ 2 \\ 1 \end{array} \right)$$

$$= \left( \begin{array}{c} 7 \\ 18 \end{array} \right)$$

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6. Determine whether the following are linear transformations from \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \).

(b) \( L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T \)
(d) \( L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)^T \)

**Answer:**

(b) Yes, \( L \) is a linear transformation. It is the same as \( L(\mathbf{x}) = A\mathbf{x} \) where

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}
\]

(d) No, \( L \) is not a linear transformation. Here is why:

\[
L\left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}
\]
\[
L\left( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix}
\]

Since

\[
L\left( \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) \neq 2L\left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right),
\]

\( L \) is not a linear transformation.

7. Determine whether the following are linear operators on \( \mathbb{R}^{m \times n} \).

(a) \( L(A) = 2A \)
(c) \( L(A) = A + I \)

**Answer:**
(a) Yes, $L$ is a linear transformation, because

$$L(\alpha A) = 2(\alpha A) = \alpha(2A) = \alpha L(A)$$

and

$$L(A + B) = 2(A + B) = 2A + 2B = L(A) + L(B)$$

(c) No, $L$ is not a linear transformation, because

$$L\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right) = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Since

$$L\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right) \neq 2L\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right),$$

$L$ is not a linear transformation.

10. For each $f \in C[0, 1]$ define $L(f) = F$, where

$$F(x) = \int_0^x f(t) \, dt \quad 0 \leq x \leq 1$$

Show that $L$ is a linear operator on $C[0, 1]$ and then find $L(e^x)$ and $L(x^2)$.

**Answer:** $L$ is a linear operator on $C[0, 1]$, because

$$L(\alpha f) = \int_0^x \alpha f(t) \, dt = \alpha \int_0^x f(t) \, dt = \alpha L(f)$$

and

$$L(f + g) = \int_0^x f(t) + g(t) \, dt = \int_0^x f(t) \, dt + \int_0^x g(t) \, dt = L(f) + L(g)$$
We can compute $L(e^x)$ and $L(x^2)$:

\[
L(e^x) = \int_0^x e^t \, dt = [e^t]_0^x = e^x - 1
\]

\[
L(x^2) = \int_0^x t^2 \, dt = \left[ \frac{t^3}{3} \right]_0^x = \frac{x^3}{3}
\]

17. Determine the kernel and the range of each of the following linear transformations on $\mathbb{R}^3$.

(a) $L(x) = (x_3, x_2, x_1)^T$

(b) $L(x) = (x_1, x_2, 0)^T$

(c) $L(x) = (x_1, x_1, x_1)^T$

**Answer:**

(a) This linear transformation is the same as multiplication by the matrix

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of $L$ is \{0\}.

The range of this linear transformation is the same as the column space of the matrix. Thus, the range of $L$ is $\mathbb{R}^3$.

(b) This linear transformation is the same as multiplication by the matrix

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of $L$ is spanned by the vector

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]
The range of this linear transformation is the same as the column space of the matrix. Thus, the range of $L$ is spanned by the vectors 
\[
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

(c) This linear transformation is the same as multiplication by the matrix 
\[
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

The kernel of this linear transformation is the same as the nullspace of the matrix. Thus, the kernel of $L$ is spanned by the vectors 
\[
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

The range of this linear transformation is the same as the column space of the matrix. Thus, the range of $L$ is spanned by the vector 
\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]

19. (a) Determine the kernel and range of the following linear operator on $P_3$:

\[L(p(x)) = xp'(x)\]

**Answer:** Suppose that $p(x) = ax^2 + bx + c$ is in the kernel of $L$.

\[L(p(x)) = 2ax^2 + bx\]

Thus, if $p(x)$ is in the kernel of $L$, $2ax^2 + bx = 0$ for all $x$, which implies that $a = 0$ and $b = 0$. Thus, every polynomial in the kernel of $L$ is of the form $p(x) = c$. Thus,

\[\ker(L) = \text{Span}(1)\]

To determine the range of $L$, we again consider an arbitrary polynomial $p(x) = ax^2 + bx + c$, and apply $L$ to the polynomial. \[L(p(x)) = 2ax^2 + bx.\] Thus, the range of $L$ is all polynomials of the form $2ax^2 + bx$. Thus,
An alternative way to find the kernel and range of $L$ is to find the matrix representing $L$ with respect to the basis $[1, x, x^2]$.

\[
L(1) = 0 \\
L(x) = x \\
L(x^2) = 2x^2
\]

Thus, the matrix representing $L$ with respect to the basis $[1, x, x^2]$ is

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

The nullspace of this matrix is the span of the vector \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), which corresponds to the polynomial 1. Thus,

\[\ker(L) = \text{Span}(1)\]

The column space of this matrix is the span of the vectors \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \), which correspond to the polynomials $x$ and $2x^2$. Thus,

\[\text{range}(L) = \text{Span}(x, 2x^2)\]

Note that \( \text{Span}(x, 2x^2) = \text{Span}(x, x^2) \) so the two answers obtained for the range of $L$ are the same.