MARKOV-BERNSTEIN TYPE INEQUALITIES FOR POLYNOMIALS UNDER ERDŐS-TYPE CONSTRAINTS

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Abstract. Throughout his life Erdős showed a particular fascination with inequalities for constrained polynomials. One of his favorite type of polynomial inequalities was Markov- and Bernstein-type inequalities. For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest. He liked to see what happened when the polynomials are restricted in certain ways. Markov- and Bernstein-type inequalities for classes of polynomials under various constraints have attracted a number of authors. In a short paper in 1940 Erdős [E40] has found a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(x)| \leq \min\left\{\frac{c\sqrt{n}}{(1-x^2)^2}\,,\;\; \frac{en}{2}\right\} \max_{t\in[-1,1]} |p(t)|\,, \qquad x \in (-1,1)\,,$$

for every polynomial p of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. See [E40]. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. The above Markov- and Bernstein-type inequalities of Erdős have been extended later in many directions. We survey a number of these inequalities under various constraints on the zeros and coefficients of the polynomials. The focus will be mainly on the directions I contributed throughout the last decade.

0. Introduction

We introduce the following classes of polynomials. Let

$$\mathcal{P}_n := \left\{ f : f(x) = \sum_{j=0}^{n} a_j x^j\,, \quad a_j \in \mathbb{R} \right\}$$

denote the set of all algebraic polynomials of degree at most n with real coefficients. Let

$$\mathcal{P}_n^c := \left\{ f : f(x) = \sum_{j=0}^{n} a_j x^j\,, \quad a_j \in \mathbb{C} \right\}$$

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denote the set of all algebraic polynomials of degree at most n with complex coefficients. Let

$$\mathcal{T}_n := \left\{ f : f(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx)\,, \quad a_j, b_j \in \mathbb{R} \right\}$$

denote the set of all trigonometric polynomials of degree at most n with real coefficients. Let

$$\mathcal{T}_n^c := \left\{ f : f(x) = a_0 + \sum_{j=0}^{n} (a_j \cos jx + b_j \sin jx)\,, \quad a_j, b_j \in \mathbb{C} \right\}$$

denote the set of all trigonometric polynomials of degree at most n with complex coefficients.

Bernstein's inequality asserts that

$$\max_{t \in [-\pi,\pi]} |p'(t)| \le n \max_{t \in [-\pi,\pi]} |p(t)|$$

for every trigonometric polynomial $p \in \mathcal{T}_n^c$. Applying this with the trigonometric polynomial $q \in \mathcal{T}_n^c$ defined by $q(t) := p(\cos t)$ with an arbitrary $p \in \mathcal{P}_n^c$, we obtain the algebraic polynomial version of Bernstein's inequality stating that

$$|p'(x)| \le \frac{n}{\sqrt{1-x^2}} \max_{u \in [-1,1]} |p(u)|\,, \qquad x \in (-1,1)\,,$$

for every polynomial $p \in \mathcal{P}_n^c$.

The inequality

$$\max_{x \in [-1,1]} |p'(x)| \le n^2 \max_{x \in [-1,1]} |p(x)|$$

for every $p \in \mathcal{P}_n^c$ is known as Markov inequality. For proofs of Bernstein's and Markov's inequalities, see [BE95a], [DL93], or [L86]. These inequalities can be extended to higher derivatives. The sharp extension of Bernstein's inequality is easy by induction, while the sharp extension of the Markov inequality requires some serious extra work. Bernstein proved the first inequality above in 1912 with $2n$ in place of n. The sharp inequality appears first in a paper of Fekete in 1916 who attributes the proof to Fejér. Bernstein attributes the proof to Landau. The inequality

$$\max_{x \in [-1,1]} |p^{(m)}(x)| \le T_n^{(m)}(1) \cdot \max_{x \in [-1,1]} |p(x)|$$

for every $p \in \mathcal{P}_n^c$ was first proved by V.A. Markov in 1892 (here T_n denotes the Chebyshev polynomial of degree n). He was the brother of the more famous A.A. Markov who proved the above inequality for $m = 1$ in 1889 by answering a question raised by the prominent Russian chemist, D. Mendeleev. See [M89]. S.N. Bernstein presented a shorter variational proof of V.A. Markov's inequality in 1938. See [B58]. The simplest known proof of Markov's inequality for higher derivatives are due to Duffin and Shaeffer [DS41], who gave various extensions as well.

Various analogues of the above two inequalities are known in which the underlying intervals, the maximum norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. These inequalities are called Markov- and Bernstein-type inequalities. If the norms are the same in both sides, the inequality is called Markov-type, otherwise it is called Bernstein-type (this distinction is not completely standard). Markov- and Bernstein-type inequalities are known on various regions of the complex plane and the n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints, exponential sums of n terms, just to mention a few. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in proving so-called inverse theorems of approximation. There are many books discussing Markov- and Bernstein-type inequalities in detail.

Throughout his life Erdős showed a particular fascination with inequalities for constrained polynomials. One of his favorite type of polynomial inequalities was Markov- and Bernstein-type inequalities. For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest. He liked to see what happened when the polynomials are restricted in certain ways. Markov- and Bernstein-type inequalities for classes of polynomials under various constraints have attracted a number of authors. For example, it has been observed by Bernstein [B58] that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. He proved that if n is odd, then

$$\sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} = \left(\frac{n+1}{2}\right)^2 ,$$

where the supremum is taken for all $0 \neq p \in \mathcal{P}_n$ that are monotone on $[-1,1]$. Here, and in what follows,

$$\|p\|_A := \sup_{x \in A} |p(x)| .$$

The above result of Bernstein may look quite surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial, then there should be a more significant improvement in the Markov inequality. In a short paper in 1940 Erdős [E40] has found a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(x)| \leq \min\left\{ \frac{c\sqrt{n}}{(1-x^2)^2},\ \frac{en}{2} \right\} \|p\|_{[-1,1]} , \qquad x \in (-1,1) , \tag{0.1}$$

for every polynomial p of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated a number of people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. The above Markov- and Bernstein-type inequalities of Erdős have been extended later in many directions.

Markov- and Bernstein-type inequalities in L_p norms are discussed, for example, in [BE95a], [DL93], [LGM96], [GL89], [N79], [MN80], [RS83], and [MMR94].

1. Markov- (and Bernstein-)type Inequalities on $[-1,1]$ for Real Polynomials with Restricted Zeros

The following result, that was anticipated by Erdős and proved in [E89] is discussed in the recent books [BE95a] and [LGM96] in a more general setting. There is an absolute constant c such that

$$\|p'\|_{[-1,1]} \le \min\left\{\frac{cn}{\sqrt{r}}\,, n^2\right\} \|p\|_{[-1,1]}$$

for every $p \in \mathcal{P}_n(r)$, where $\mathcal{P}_n(r)$ denotes the set of all polynomials of degree at most n with *real coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \le 1)$. Another kind of essentially sharp extension of Erdős' inequality is proved in [BE94] and partially discussed in the books [BE95a] and [MMR94]. It states that there is an absolute constant $c > 0$ such that

$$|p'(x)| \le c \min\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\ \ n(k+1)\right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all polynomials $p \in \mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ denotes the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk.

The history of the this result is briefly the following. After a number of less general and weaker results of Erdős [E40], Lorentz [L63], Scheick [Sch72], Szabados and Varma [SzV80], Szabados [Sz81], and Máté [M81], the essentially sharp Markov-type estimate

$$c_1 n(k+1) \le \sup_{p \in \mathcal{P}_{n,k}} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \le c_2 n(k+1) \tag{1.1}$$

was proved by P. Borwein [B85] in a slightly less general formulation. The above form of the result appeared in [E87a] first. Here $c_1 > 0$ and $c_2 > 0$ are absolute constants. A simpler proof of the upper bound of (1.1) is given in [E91a] that relates the upper bound in (1.1) to a beautiful Markov-type inequality of Newman [N76] (see Theorem 5.1 later in this paper) for Müntz polynomials. See also [BE95a] and [LGM96]. A sharp extension of (1.1) to L_p norms is proved in [BE95d]. The lower bound in (1.1) was proved and the upper bound was conjectured by Szabados [Sz81] earlier. Another example that shows the lower bound in (1.1) is given in [E87b].

Erdős [E40] proved the (Markov-)Bernstein-type inequality (0.1) on $[-1,1]$ for polynomials from $\mathcal{P}_{n,0}$ having only real zeros. Lorentz [L63] improved this by establishing the "right" Bernstein-type inequality on $[-1,1]$ for all polynomials from $\mathcal{P}_{n,0}$. Improving weaker results of [E87b] and [ESz89b], in [BE94] we obtained a Bernstein-type analogue of the upper bound in (1.1) which was believed to be essentially sharp. Namely we proved

$$\sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c \min\{B_{n,k,x}, M_{n,k}\} \tag{1.2}$$

for every $x \in (-1,1)$, where

$$B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,, \qquad \text{and} \qquad M_{n,k} := n(k+1)\,,$$

and where $c > 0$ is an absolute constant. Although it was expected that this is the "right" Bernstein-type inequality for the classes $\mathcal{P}_{n,k}$, its sharpness was proved only in the special cases when $x = 0$ or $x = \pm 1$; when $k = 0$; and when $k = n$. See [E87a], [E87b], and [E90a]. The sharpness of (1.2) is shown in [E98b]. Summarizing the results of [BE94] and [E98b], we have

$$c_1 \min\{B_{n,k,x}, M_{n,k}\} \le \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 \min\{B_{n,k,x}, M_{n,k}\}\,,$$

for every $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants.

2. Markov- and Bernstein-type Inequalities on $[-1,1]$ for Complex Polynomials with Restricted Zeros

As before, let $\mathcal{P}_n(r)$ be the set of all polynomials of degree at most n with *real coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \le 1)$. Let $\mathcal{P}_n^c(r)$ be the set of all polynomials of degree at most n with *complex coefficients* and with no zeros in the union of open disks with diameters $[-1,-1+2r]$ and $[1-2r,1]$, respectively $(0 < r \le 1)$.

Essentially sharp Markov-type inequalities for $\mathcal{P}_n^c(r)$ on $[-1,1]$ are established in [E98a], where the inequalities

$$c_1 \min\left\{\frac{n\log\left(e+n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\} \le \sup_{p \in \mathcal{P}_n^c(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \le c_2 \min\left\{\frac{n\log\left(e+n\sqrt{r}\right)}{\sqrt{r}}\,,\, n^2\right\}$$

are established for every $0 < r \le 1$ with absolute constants $c_1 > 0$ and $c_2 > 0$. This result should be compared with the inequalities

$$c_1 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\} \le \sup_{p \in \mathcal{P}_n(r)} \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \le c_2 \min\left\{\frac{n}{\sqrt{r}}\,,\, n^2\right\}\,,$$

for every $0 < r \le 1$ with absolute constants $c_1 > 0$ and $c_2 > 0$. See [E89] and [LGM96].

As before, let $\mathcal{P}_{n,k}$ denote the set of all polynomials of degree at most n with *real coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Let $\mathcal{P}_{n,k}^c$ denote the set of all polynomials of degree at most n with *complex coefficients* and with at most k $(0 \le k \le n)$ zeros in the open unit disk. Associated with $0 \le k \le n$ and $x \in (-1,1)$, let

$$B_{n,k,x}^* := \max\left\{\sqrt{\frac{n(k+1)}{1-x^2}}\,,\;\; n\log\left(\frac{e}{1-x^2}\right)\right\}\,, \qquad B_{n,k,x} := \sqrt{\frac{n(k+1)}{1-x^2}}\,,$$

and

$$M_{n,k}^* := \max\{n(k+1),\;\; n\log n\}\,, \qquad M_{n,k} := n(k+1)\,.$$

In [E98a] and [E98b] it is shown that

$$c_1 \min\{B_{n,k,x}^*, M_{n,k}^*\} \le \sup_{p \in \mathcal{P}_{n,k}^c} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 \min\{B_{n,k,x}^*, M_{n,k}^*\}$$

for every $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. This result should be compared with the inequalities

$$c_1 \min\{B_{n,k,x}, M_{n,k}\} \le \sup_{p \in \mathcal{P}_{n,k}} \frac{|p'(x)|}{\|p\|_{[-1,1]}} \le c_2 \min\{B_{n,k,x}, M_{n,k}\}$$

for every $x \in (-1,1)$, where $c_1 > 0$ and $c_2 > 0$ are absolute constants. See [E94] and [E98b]. It may be surprising that there is a significant difference between the real and complex cases as far as Markov-Bernstein type inequalities are concerned.

3. Lorentz Representation and Lorentz Degree

An elementary, but very useful tool for proving inequalities for polynomials with restricted zeros is the *Bernstein* or *Lorentz representation* of polynomials. Namely, each polynomial $p \in \mathcal{P}_n$ with no zeros in the open unit disk is of the form

$$p(x) = \sum_{j=0}^{d} a_j (1-x)^j (1+x)^{d-j}\,, \qquad a_j \ge 0\,, \quad j = 0, 1, \ldots, d\,, \tag{3.1}$$

with $d = n$. Moreover, if a polynomial $p \in \mathcal{P}_n$ has no zeros in the ellipse L_ε with large axis $[-1,1]$ and small axis $[-\varepsilon i, \varepsilon i]$ ($\varepsilon \in [-1,1]$) then it has a Lorentz representation (3.1) with $d \le 3n\varepsilon^{-2}$. See [ESz88]. We can combine this with the Markov-Bernstein-type inequality of Lorentz [L63], which states that there is an absolute constant $c > 0$ such that

$$|p'(x)| \le c \min\left\{ \frac{\sqrt{d}}{\sqrt{1-x^2}}\,,\ d \right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all polynomials of form (3.1) above. We obtain that there is an absolute constant $c > 0$ such that

$$|p'(x)| \le c \min\left\{ \frac{\sqrt{n}}{\varepsilon\sqrt{1-x^2}}\,,\ \frac{n}{\varepsilon^2} \right\} \|p\|_{[-1,1]}\,, \qquad x \in (-1,1)\,,$$

for all polynomials $p \in \mathcal{P}_n$ having no zeros in L_ε.

The minimal $d \in \mathbb{N}$ for which a polynomial p has a representation (3.1) is called the *Lorentz degree* of the polynomial and it is denoted by $d(p)$. It is easy to observe, see [ESz88], that $d(p) < \infty$ if and only if p has no zeros in $(-1,1)$. This is a theorem ascribed to Hausdorff. One of the attractive, nontrivial facts is that if

$$p(x) = ((x-a)^2 + \varepsilon^2(1-a^2))^n, \qquad 0 < \varepsilon \le 1, \quad -1 < a < 1\,,$$

then

$$c_1 n\varepsilon^{-2} \le d(p) \le c_2 n\varepsilon^{-2}$$

with absolute constants $c_1 > 0$ and $c_2 > 0$. See [E91b]. The slightly surprising fact that $d(pq) < \max\{d(p), d(q)\}$ is possible is observed in [E91b].

4. Further Markov- and Bernstein-Type Inequalities on $[-1,1]$ for Polynomials with Restricted Zeros

As in Section 3, let L_ε be the ellipse with large axis $[-1,1]$ and small axis $[-\varepsilon i, \varepsilon i]$ ($\varepsilon \in [-1,1]$). In [ESzl we proved the essentially sharp Markov-type inequality

$$c_1 \min\left\{\frac{n}{\varepsilon}, n^2\right\} \leq \sup_p \frac{\|p'\|_{[0,1]}}{\|p\|_{[-1,1]}} \leq c_2 \min\left\{\frac{n}{\varepsilon}, n^2\right\},$$

where the supremum is taken for all polynomials $p \in \mathcal{P}_n$ having no zeros in L_ε ($c_1 > 0$ and $c_2 > 0$ are absolute constants). We also proved the essentially sharp Markov-type inequality

$$c_1 \min\left\{\frac{n}{\varepsilon} \log(e+n\varepsilon), n^2\right\} \leq \sup_p \frac{\|p'\|_{[0,1]}}{\|p\|_{[-1,1]}} \leq c_2 \min\left\{\frac{n}{\varepsilon} \log(e+n\varepsilon), n^2\right\},$$

where the supremum is taken for all polynomials $p \in \mathcal{P}_n^c$ having no zeros in L_ε ($c_1 > 0$ and $c_2 > 0$ are absolute constants). See [ESz].

For $x \in [-1+\varepsilon^2, 1-\varepsilon^2]$ we have also established essentially sharp Bernstein-type inequalities for all polynomials $p \in \mathcal{P}_n$ having no zeros in L_ε. Namely if $x \in [-1+\varepsilon^2, 1-\varepsilon^2]$, then

$$\frac{c_1}{\sqrt{1-x^2}} \min\left\{\frac{\sqrt{n}}{\sqrt{\varepsilon}}, n\right\} \leq \sup_p \frac{|p'(x)|}{\|p\|_{[-1,1]}} \leq \frac{c_2}{\sqrt{1-x^2}} \min\left\{\frac{\sqrt{n}}{\sqrt{\varepsilon}}, n\right\},$$

where the supremum is taken for all polynomials $p \in \mathcal{P}_n$ having no zeros in L_ε ($c_1 > 0$ and $c_2 > 0$ are absolute constants). See [ESz].

Note that the angle between the ellipse L_ε and the interval $[-1,1]$, as well as the angle between the unit disk D and the interval $[-1,1]$, is $\pi/2$.

Let K_α be the open diamond of the complex plane with diagonals $[-1,1]$ and $[-ia, ia]$ such that the angle between $[ia, 1]$ and $[1, -ia]$ is $\alpha\pi$.

An old question of Erdős that Halász answered recently is that how large the quantity

$$\frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}}$$

can be assuming that $p \in \mathcal{P}_n$ (or $p \in \mathcal{P}_n^c$) has no zeros in a diamond K_α, $\alpha \in [0,1)$. Halász [H] proved that there are constants $c_1 > 0$ and $c_2 > 0$ depending only on $\alpha \in [0,1)$ such that

$$c_1 n^{2-\alpha} \leq \sup_p \frac{|p'(1)|}{\|p\|_{[-1,1]}} \leq \sup_p \frac{\|p'\|_{[-1,1]}}{\|p\|_{[-1,1]}} \leq c_2 n^{2-\alpha},$$

where the supremum is taken for all polynomials $p \in \mathcal{P}_n$ or $p \in \mathcal{P}_n^c$ having no zeros in K_α.

Halász's result extends a theorem of Szegő. Let Γ be a curve and z_0 be a point of the complex plane. In 1925 Szegő examined how large the quantity

$$\omega_n := \sup_{p \in \mathcal{P}_n^c} \frac{|p'(z_0)|}{\|p\|_\Gamma}$$

can be. Suppose Γ is a closed curve with an angle $\alpha\pi$ at z_0 $(0 \le \alpha < 2)$. Then there are constants $A > 0$ and $B > 0$ depending only on the curve Γ such that

$$Bn^{2-\alpha} \le \omega_n \le An^{2-\alpha}\,.$$

If $\alpha = 2$, the inequality $\omega_n \le K \log n$ still holds with an absolute constant K. See [Sz25].

In [E90b] Erdős studies the following question. Let

$$p_n(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0\,.$$

Assume that the set

$$E(p_n) := \{z \in \mathbb{C} : |p_n(z)| \le 1\}$$

is connected. Is it true that $\|p_n'\|_{E(p_n)} \le (1/2 + o(1))n^2$? Pommerenke proved that $\|p_n'\|_{E(p_n)} \le en^2$. Erdős [E90b] speculates that the extremal case may be achieved by a linear transformation of the Chebyshev polynomial T_n.

There are several more challenging open problems about Markov- and Bernstein-type inequalities for polynomials with restricted zeros. One of these is the following.

Problem 4.1. *Is there an absolute constant c so that*

$$\sup_p \frac{|p'(1)|}{\|p\|_{[0,1]}} \le cnm\,,$$

where the supremum is taken over all polynomials $p \in \mathcal{P}_n^c$ having at most m distinct zeros (possibly of higher multiplicity)?

5. Newman's Inequality

Let $\Lambda := (\lambda_j)_{j=0}^{\infty}$ be a sequence of distinct real numbers. The linear span of

$$\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}$$

over $\mathbb{R}$ will be denoted by

$$M_n(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\}\,.$$

Elements of $M_n(\Lambda)$ are called Müntz polynomials.

Newman's inequality [N76] is an essentially sharp Markov-type inequality for $M_n(\Lambda)$, where $\Lambda := (\lambda_j)_{j=0}^{\infty}$ is a sequence of distinct nonnegative real numbers.

Newman's inequality (see [N76] and [BE95a]) asserts the following.

Theorem 5.1. *We have*

$$\frac{2}{3}\left(\sum_{j=0}^{n}\lambda_j\right) \le \sup_{0\ne p\in M_n(\Lambda)} \frac{\|xp'(x)\|_{[0,1]}}{\|p\|_{[0,1]}} \le 11\left(\sum_{j=0}^{n}\lambda_j\right).$$

Frappier [F82] shows that the constant 11 in Newman's inequality can be replaced by 8.29. In [BE96b] and [BE95a], by modifying (and simplifying) Newman's arguments, we showed that the constant 11 in the above inequality can be replaced by 9. But more importantly, this modification allowed us to prove the "right" $L_p[0,1]$ version of Newman's inequality [BE96b] (an L_2 version of which was proved earlier in [BEZ94a]).

On the basis of considerable computation, in [BE96b] we speculate that the best possible constant in Newman's inequality is 4. (We remark that an incorrect argument exists in the literature claiming that the best possible constant in Newman's inequality is at least $4+\sqrt{15}=7.87\ldots$.)

In [BE96d] we proved the following Newman-type inequality on intervals $[a,b]\subset(0,\infty)$.

Theorem 5.2. *Suppose* $\lambda_0 = 0$ *and there exists a* $\delta > 0$ *so that* $\lambda_j \ge \delta j$ *for each* j. *Suppose* $0 < a < b$. *Then there exists a constant* $c(a,b,\delta)$ *depending only on* a, b, *and* δ *so that*

$$\|p'\|_{[a,b]} \le c(a,b,\delta)\left(\sum_{j=0}^{n}\lambda_j\right)\|p\|_{[a,b]}$$

for every $p\in M_n(\Lambda)$.

Theorem 5.2 complements Newman's Markov-type inequality. In [E] Theorem 5.2 is extended to $L_p[a,b]$ norms for $1\le p\le\infty$.

Note that with the substitution $x=e^t$, Müntz polynomials turn to exponential sums. The rational functions and exponential sums belong to those concrete families of functions which are the most frequently used in nonlinear approximation theory. See, for example, Braess [B86]. The starting point of consideration of exponential sums is an approximation problem often encountered for the analysis of decay processes in natural sciences. A given empirical function on a real interval is to be approximated by sums of the form

$$\sum_{j=1}^{n} a_j e^{\lambda_j t},$$

where the parameters a_j and λ_j are to be determined, while n is fixed.

In [BE96c] we proved the "right" Bernstein-type inequality for exponential sums. This inequality is the key to proving inverse theorems for approximation by exponential sums. Let

$$E_n := \left\{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j,\lambda_j\in\mathbb{R}\right\}.$$

So E_n is the collection of all $n+1$ term exponential sums with constant first term. Schmidt [Sch70] proved that there is a constant $c(n)$ depending only on n so that

$$\|f'\|_{[a+\delta,b-\delta]} \le c(n)\delta^{-1}\|f\|_{[a,b]}$$

for every $f \in E_n$ and $\delta \in \big(0, \frac{1}{2}(b-a)\big)$. Lorentz [L89] improved Schmidt's result by showing that for every $\alpha > \frac{1}{2}$, there is a constant $c(\alpha)$ depending only on α so that $c(n)$ in the above inequality can be replaced by $c(\alpha)n^{\alpha \log n}$ (Xu improved this to allow $\alpha = \frac{1}{2}$), and he speculated that there may be an absolute constant c so that Schmidt's inequality holds with $c(n)$ replaced by cn. We [BE95c] proved a weaker version of this conjecture with cn^3 instead of cn. The main result of [BE96c] (see also [BE95a] shows that Schmidt's inequality holds with $c(n) = 2n-1$. This essentially sharp result can also be formulated as

Theorem 5.3. *We have*

$$\frac{1}{e-1}\,\frac{n-1}{\min\{x-a, b-x\}} \le \sup_{0 \ne f \in E_n} \frac{|f'(x)|}{\|f\|_{[a,b]}} \le \frac{2n-1}{\min\{x-a, b-x\}}$$

for all $x \in (a,b)$.

An L_p version of Theorem 5.3 is established in [E].

6. Bernstein's Inequality on the Unit Disk

We use the notation

$$D := \{z \in \mathbb{C} : |z| \le 1\}, \qquad \text{and} \qquad \partial D := \{z \in \mathbb{C} : |z| = 1\}$$

The classical inequalities of Bernstein state that

$$|p'(z_0)| \le n\,\|p\|_{\partial D} \tag{6.1}$$

for every $p \in \mathcal{P}_n^c$ and $z_0 \in \partial D$;

$$|t'(\theta_0)| \le n\,\|t\|_K \tag{6.2}$$

for every $t \in \mathcal{T}_n^c$ and $\theta_0 \in K$;

$$|p'(x_0)| \le \frac{n}{\sqrt{1-x_0^2}}\,\|p\|_{[-1,1]} \tag{6.3}$$

for every $p \in \mathcal{P}_n^c$ and $x_0 \in (-1,1)$. Proofs of the above inequalities may be found in almost every book on approximation theory. See [DL93], [BE95a], or [L86], for instance.

It was conjectured by Erdős and proved by P. Lax that

$$|p'(z_0)| \le \frac{n}{2}\,\|p\|_{\partial D} \tag{6.4}$$

for every $p \in \mathcal{P}_n^c$ having no zeros in D and for every $z_0 \in \partial D$.

We define the rational function space

$$\mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D) := \left\{ \frac{p_n(z)}{\prod_{j=1}^{n}(z-a_j)} : p_n \in \mathcal{P}_n^c \right\}$$

on ∂D with $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$. In [BE96a] the following pair of theorems is proved The first one may be viewed as an extension of Bernstein's inequality (6.1), while the second one may be viewed as an extension of Lax's inequality (6.4) (in both cases we let each $|a_j|$ tend to infinity.

Theorem 6.1. *Let $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \partial D$. Then*

$$|f'(z_0)| \le \max \left\{ \sum_{\substack{j=1 \\ |a_j|>1}} \frac{|a_j|^2 - 1}{|a_j - z_0|^2}, \sum_{\substack{j=1 \\ |a_j|<1}} \frac{1 - |a_j|^2}{|a_j - z_0|^2} \right\} \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$ and $z_0 \in \partial D$. If the first sum is not less than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c\, S_n^+$, $c \in \mathbb{C}$, where S_n^+ is the Blaschke product associated with those a_j for which $|a_j| > 1$. If the first sum is not greater than the second sum for a fixed $z_0 \in \partial D$, then equality holds for $f = c\, S_n^-$, $c \in \mathbb{C}$, where S_n^- is the Blaschke product associated with those a_j for which $|a_j| < 1$.

Theorem 6.2. *Let $\{a_1, a_2, \cdots, a_n\} \subset \mathbb{C} \setminus \overline{D}$. Then*

$$|f'(z_0)| \le \frac{1}{2} \left(\sum_{j=1}^{n} \frac{|a_j|^2 - 1}{|a_j - z_0|^2} \right) \|f\|_{\partial D}$$

for every $f \in \mathcal{P}_n^c(a_1, a_2, \cdots, a_n; \partial D)$ having no zeros in D and for every $z_0 \in \partial D$. Equality holds for $h = c\,(S_n + 1)$ with $c \in \mathbb{C}$, where S_n is the Blaschke product associated with $(a_k)_{k=1}^n$.

In [BEZ94b] and [BE96a] we proved a number of inequalities for rational function spaces. For example, the sharp Bernstein-type inequality

$$|f'(y)| \le \left(\frac{1}{\sqrt{1 - y^2}} \operatorname{Re} \left(\sum_{k=1}^{n} \frac{\sqrt{a_k^2 - 1}}{a_k - y} \right) \right) \|f\|_{[-1,1]}\,, \qquad y \in (-1, 1)\,,$$

is proved for all rational functions $f = p/q$, where p is a polynomial of degree at most n with real coefficients,

$$q(x) = \prod_{k=1}^{n} |x - a_k|\,, \qquad a_k \in \mathbb{C} \setminus [-1, 1]\,,$$

and the square roots are defined so that

$$\left| a_k - \sqrt{a_k^2 - 1} \right| < 1\,, \qquad k = 1, 2, \ldots, n\,.$$

When the poles $a_1, a_2, \ldots, a_n$ are real, the proof relies on the explicit computation of the Chebyshev "polynomials" for the Chebyshev space

$$\operatorname{span} \left\{ 1\,, \frac{1}{x - a_1}\,, \ldots\,, \frac{1}{x - a_n} \right\}.$$

7. Markov-type Inequalities on $[0,1]$ under Littlewood-type Coefficient Constraints

Erdős studied and raised many questions about polynomials with restricted coefficients. Both Erdős and Littlewood showed particular fascination about the class $\mathcal{L}_n$, where $\mathcal{L}_n$ denotes the set of all polynomials of degree n with each of their coefficients in $\{-1,1\}$. A related class of polynomials is $\mathcal{F}_n$ that denotes the set of all polynomials of degree at most n with each of their coefficients in $\{-1,0,1\}$. Another related class is $\mathcal{G}_n$, that is the collection of all polynomials p of the form

$$p(x)=\sum_{j=m}^{n} a_j x^j\,, \qquad |a_m|=1\,, \quad |a_j|\le 1\,,$$

where m is an unspecified nonnegative integer not greater than n.

In [BE97] and [BE] we establish the right Markov-type inequalities for the classes $\mathcal{F}_n$ and $\mathcal{G}_n$ on $[0,1]$. Namely there are absolute constants $c_1>0$ and $c_2>0$ such that

$$c_1 n\log(n+1)\le \max_{0\ne p\in\mathcal{F}_n}\frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}}\le c_2 n\log(n+1)$$

and

$$c_1 n^{3/2}\le \max_{0\ne p\in\mathcal{G}_n}\frac{|p'(1)|}{\|p\|_{[0,1]}}\le \max_{0\ne p\in\mathcal{G}_n}\frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}}\le c_2 n^{3/2}\,.$$

It is quite remarkable that the right Markov factor for $\mathcal{G}_n$ is much larger than the right Markov factor for $\mathcal{F}_n$. In [BE] we also show that there are absolute constants $c_1>0$ and $c_2>0$ such that

$$c_1 n\log(n+1)\le \max_{0\ne p\in\mathcal{L}_n}\frac{|p'(1)|}{\|p\|_{[0,1]}}\le \max_{0\ne p\in\mathcal{L}_n}\frac{\|p'\|_{[0,1]}}{\|p\|_{[0,1]}}\le c_2 n\log(n+1)$$

for every $p\in\mathcal{L}_n$. For polynomials $p\in\mathcal{F}:=\bigcup_{n=0}^{\infty}\mathcal{F}_n$ with $|p(0)|=1$ and for $y\in[0,1)$ the Bernstein-type inequality

$$\frac{c_1\log\left(\frac{2}{1-y}\right)}{1-y}\le \max_{\substack{p\in\mathcal{F}\\ |p(0)|=1}}\frac{\|p'\|_{[0,y]}}{\|p\|_{[0,1]}}\le \frac{c_2\log\left(\frac{2}{1-y}\right)}{1-y}$$

is also proved with absolute constants $c_1>0$ and $c_2>0$.

8. Markov- and Bernstein-Type Inequalities for Self-reciprocal and Anti-self-reciprocal Polynomials

Let SR_n^c denote the set of all self-reciprocal polynomials $p_n\in\mathcal{P}_n^c$ satisfying

$$p_n(z)=z^n p_n(z^{-1})\,.$$

Let SR_n denote the set of all real self-reciprocal polynomials of degree at most n, that is, $\mathrm{SR}_n := \mathrm{SR}_n^c \cap \mathcal{P}_n$. For a polynomial $p_n \in \mathcal{P}_n^c$ of the form

$$p_n(z) = \sum_{j=0}^{n} c_j z^j \,, \qquad c_j \in \mathbb{C}\,, \tag{8.1}$$

$p_n \in \mathrm{SR}_n^c$ if and only if

$$c_j = c_{n-j}\,, \qquad j = 0, 1, \dots, n\,.$$

Let ASR_n^c denote the set of all anti-self-reciprocal polynomials $p_n \in \mathcal{P}_n^c$ satisfying

$$p_n(z) = -z^n p_n(z^{-1})\,.$$

Let ASR_n denote the set of all real anti-self-reciprocal polynomials, that is, $\mathrm{ASR}_n := \mathrm{ASR}_n^c \cap \mathcal{P}_n$. For a polynomial $p \in \mathcal{P}_n^c$ of the form (8.1), $p_n \in \mathrm{ASR}_n^c$ if and only if

$$c_j = -c_{n-j}\,, \qquad j = 0, 1, \dots, n\,.$$

Every $p_n \in \mathrm{SR}_n^c$ and $p_n \in \mathrm{ASR}_n^c$ satisfies the growth condition

$$|p_n(x)| \le (1 + |x|^n)\|p_n\|_{[-1,1]}\,, \qquad x \in \mathbb{R}\,. \tag{8.2}$$

The Markov-type (uniform) part of the following inequality is due to Kroó and Szabados [KSz94]. For the Bernstein-type (pointwise) part, see [BE95a].

Theorem 8.1. *There is an absolute constant $c_1 > 0$ such that*

$$|p_n'(x)| \le c_1 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\} \|p_n\|_{[-1,1]}$$

for every $x \in (-1,1)$ and for every polynomial $p_n \in \mathcal{P}_n^c$ satisfying the growth condition (8.2), in particular for every $p_n \in \mathrm{SR}_n^c$ and for every $p_n \in \mathrm{ASR}_n^c$ $(n \ge 2)$.

It is shown in [BE95a] that the above result is sharp for the classes SR_n and ASR_n, that is, there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\} \le \sup_{p_n} \frac{|p_n'(x)|}{\|p_n\|_{[-1,1]}} \le c_2 n \min\left\{\log n\,,\ \log\left(\frac{e}{1-x^2}\right)\right\}\,,$$

where the supremum is taken either for all $0 \ne p_n \in \mathrm{SR}_n$ or for all $0 \ne p_n \in \mathrm{ASR}_n$ $(n \ge 2)$. Associated with a polynomial $p_n \in \mathcal{P}_n^c$ of the form (8.1) we define the polynomial

$$p_n^*(z) = \sum_{j=0}^{n} \overline{c}_{n-j} z^j\,.$$

It was proved by Malik [MMR94] that

$$\max_{z \in \partial D} \left(|p_n'(z)| + |{p_n^*}'(z)|\right) \le n \max_{z \in \partial D} |p_n(z)|\,.$$

In particular, if $p_n \in \mathcal{P}_n^c$ is conjugate reciprocal (satisfying $p_n = p_n^*$), then

$$\max_{z\in\partial D} |p_n'(z)| \leq \frac{n}{2} \max_{z\in\partial D} |p_n(z)|\,.$$

In [RS83] the inequality

$$\max_{z\in\partial D} |p_n'(z)| \leq (n-1/4) \max_{z\in\partial D} |p_n(z)|\,.$$

is stated for all $p_n \in \mathrm{SR}_n^c$ and a slightly better bound is proved by Frappier, Rahman, and Ruscheweyh [FRR85]. They also show that in this inequality the Bernstein factor $(n-1/4)$, in general, cannot be replaced by anything better than $(n-1)$.

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