Problems to Introduction to Real Analysis, (Math446)

Due: 09/09/04

Prof.: Thomas Schlumprecht

Problem 1. For two non empty sets $A, B \subset \mathbb{R}$, show that
\[
\inf(A + B) = \inf A + \inf B,
\]
where $A + B := \{a + b : a \in A, b \in B\}$.

Problem 2. Assume that $(a_n)$ and $(b_n)$ are two sequences and that
\[
a = \lim_{n \to \infty} a_n \text{ and } b = \lim_{n \to \infty} b_n
\]
exist. Show that
\[
a \cdot b = \lim_{n \to \infty} a_n b_n.
\]

Problem 3. Assume that $f : \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ are continuous. Show that $f + g$ and $f \circ g$ (composition) is continuous.

Problem 4. For a sequence $(x_n) \subset \mathbb{R}$ we define
\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup_{k \geq n} x_k,
\]
\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf_{k \geq n} x_k,
\]
a) State the theorem (without proof) which ensures that above limits exist (possibly $\pm\infty$) , and show that this theorem is applicable.
b) For two sequences $(x_n), (y_n) \subset \mathbb{R}$ show that
\[
\limsup_{n \to \infty} x_n + y_n \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.
\]
c) Find two sequences $(x_n), (y_n) \subset \mathbb{R}$ for which
\[
\limsup_{n \to \infty} x_n + y_n \neq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.
\]

Problem 5. Assume that for a sequence $(a_n)$ $\limsup_{n \to \infty} a_n = a$. Show that there is a subsequence of $(a_n)$ which converges to $a$. 
Problem 6. Construction of the rational numbers \( \mathbb{Q} \) starting with the integers \( \mathbb{Z} \).

On \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \) we define the following relation:

\[ (p, q) \sim (p', q') \iff pq' = p'q. \]

Show that \( \sim \) is an equivalence relation on \( \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \).

Problem 7. We define \( \mathbb{Q} \) to be all the equivalence classes of \( \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \) with respect to \( \sim \), i.e.,

\[ \mathbb{Q} = \{(p, q) : p, q \in \mathbb{Z}, q \neq 0 \}, \text{ where} \]

\[ (p, q) = \{(p', q') : p', q' \in \mathbb{Z}, q' \neq 0, \text{ and } p'q = pq' \}. \]

From now on we write \( \frac{p}{q} \) instead of \( (p, q) \) (which is more suggestive).

We define the following operations on \( \mathbb{Q} \):

\[ \frac{p}{q} + \frac{s}{t} = \frac{pt + sq}{qt} \quad \text{and} \quad \frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}. \]

Show that these operations are welldefined, and among the axioms of a field, show that \( \mathbb{Q} \) verify five of them (any five you want).

Problem 8. For \( \frac{p}{q}, \frac{s}{t} \in \mathbb{Q} \) we define:

\[ \frac{p}{q} < \frac{s}{t} \iff pt < qs. \]

Show that \( \mathbb{Q} \) is an ordered field.