Problem 1. Let $X$ be a vectorspace over $\mathbb{R}$. We call a map $\| \cdot \| : X \rightarrow [0, \infty)$ a norm on $X$ if the following properties are satisfied:

a) For all $x \in X$: $\|x\| = 0 \iff x = 0$,
b) for all $x \in X$ and $\alpha \in \mathbb{R}$: $\|\alpha x\| = |\alpha| \cdot \|x\|$,
c) for all $x, y \in X$: $\|x + y\| \leq \|x + y\|$.

In that case we call $(X, \| \cdot \|)$ a normed linear space or normed vectorspace. Show that the map:

$$d : X \times X \rightarrow [0, \infty), \quad (x, y) \mapsto d(x, y) = \|x - y\|,$$

is a metric.

Problem 2. For $f, g \in C([0, 1])$:

$$d_1(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \quad \text{and} \quad d_2(f, g) = \int_0^1 |f(x) - g(x)| dx.$$

a) Show that $d_1$ and $d_2$ are metrics, and show that $d_1$ and $d_2$, come from a norm as in Problem 1.
b) Show that: If $(f_n)$ is convergent with respect to $d_1$ then it is also convergent with respect to $d_2$.
c) Show that there is a sequence $(f_n)$ which converges to 0 for $d_2$ but not for $d_1$.

Problem 3. Let $(X, d)$ be a metric space. For $A \subset X$ define

$$A^\circ = \bigcup \{ U : U \subset A \text{ open } \} \quad \text{and} \quad \overline{A} = \bigcup \{ F : F \supset A \text{ closed } \}.$$

$A^\circ$ is called open kernel of $A$ and $\overline{A}$ is called closure of $A$.

a) $A^\circ$ is open,
b) $\overline{A}$ is closed,
c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
d) $\overline{A} = A \cup \{ x : x \text{ is limit point of } A \}$.

Problem 4. Assume that $(X, d)$ is a metric space. Denote the open sets in $X$ by $T_d$ and the closed sets by $F_d$. Show that

a) $T_d$ is closed under taking arbitrary unions, and finite intersections,
b) $F_d$ is closed under arbitrary intersections, and finite unions.

Problem 5. On a test a student was asked to write down the property of a function $d : X \times X \rightarrow \mathbb{R}$, which turns $d$ into a metric. She wrote down the following conditions $(x, y, z \in X \text{ arbitrary})$:
(i) \( d(x, y) = 0 \iff x = y \),
(ii) \( d(x, y) = d(y, x) \),
(iii) \( d(x, z) \leq d(x, y) + d(y, z) \).

The professor deduced 2 points. Why?
After class she goes to the professor and asserts that these three properties imply that \( d \) is a metric. Is she right?

**Problem 6.** Assume that \((X, d)\) is a metric space. Denote the open sets in \( X \) by \( \mathcal{T}_d \) and the closed sets by \( \mathcal{F}_d \). Show that
a) \( \mathcal{T}_d \) is closed under taking arbitrary unions, and finite intersections,
b) \( \mathcal{F}_d \) is closed under arbitrary intersections, and finite unions.

**Problem 7.** Let \((x_i)_{i \in I}\) and \((y_i)_{i \in I}\) be two convergent nets \(((I, \leq) \text{ directed partial order})\) in a normed linear space \( X \). Show that:
\[
\lim_{i \in I} x_i + y_i = \lim_{i \in I} x_i + \lim_{i \in I} y_i.
\]

**Problem 8.**

a) Let \( A \subset [0, 1] \) be finite and define for \( x \in [0, 1] \)
\[
f(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \not\in A 
\end{cases}.
\]

Show that \( f \) is Riemann integrable and compute its integral.

b) Define for \( x \in [0, 1] \)
\[
f(x) = \begin{cases} 
1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\
0 & \text{otherwise.}
\end{cases}
\]

Show that \( f \) is Riemann integrable and compute its integral.

c) Define for \( x \in [0, 1] \)
\[
f(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{otherwise.}
\end{cases}
\]

Show that \( f \) is not Riemann integrable.