Problems to Introduction to Real Analysis, (Math446)  
Due: 09/16/02

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Problem 1. Every finite subset of $\mathbb{R}$ is closed.

Problem 2. Find a sequence of open sets $(U_n)$ and sequence of closed sets $(V_n)$ so that

$\bigcap_{n \in \mathbb{N}} U_n$ is not open,

$\bigcup_{n \in \mathbb{N}} V_n$ is not closed.

Problem 3. Using the “finite covering property” show that the set

$\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

is compact and that the set

$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

is not compact.

Problem 4. Let $A \subseteq \mathbb{R}$. We call the set

$\overline{A} = \bigcap\{B : A \subset B \subset \mathbb{R}, B \text{ closed}\}$,

the closure of $A$, and the set

$A^\circ = \bigcup\{U : U \subset A, U \text{ open}\}$,

the open interior of $A$. Show that

a) $\overline{A}$ is closed and that $A$ is closed if and only if $\overline{A} = A$.

b) $A^\circ$ is open that $A$ is open if and only if $A^\circ = A$.

Problem 5. Determine (with proof) the closure and the interior of the following sets: $[0, 1) \cup \{2\}$, $\mathbb{N}$, $\mathbb{Q}$.

Show that for $A, B \subseteq \mathbb{R}$

$\overline{A \cap B} \subset \overline{A} \cup \overline{B}$,

and that equality does not necessarily hold.

Problem 5. Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then $a$ is a limit point of $A$ if and only if there is sequence $(a_n) \subset A$ which converges to $a$.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the image of a compact set $K \subseteq \mathbb{R}$ under $f$ (i.e., $f(K) = \{f(x) : x \in K\}$) is compact.

Show by examples that the image of open/closed sets under continuous maps $f : \mathbb{R} \to \mathbb{R}$ do not need to be open/closed.