1. Find the volume of the solid under \( z = 2x^2y \) above the region in the \( xy \)-plane between \( y = x \) and \( y = x^2 \).
   
   a. \( \frac{2}{35} \)  
   b. \( \frac{35}{12} \)  
   c. \( \frac{12}{35} \)  
   d. \( \frac{1}{35} \)  
   e. \( \frac{1}{12} \)

   \[
   V = \iint 2x^2y \, dA = \int_0^1 \int_{x^2}^x 2x^2 y \, dy \, dx = \int_0^1 \left[ x^2 y^2 \right]_{x^2}^x dx = \int_0^1 (x^4 - x^6) \, dx = \left[ \frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 = \frac{1}{5} - \frac{1}{7} = \frac{7 - 5}{35} = \frac{2}{35}
   \]

2. Compute \( \iint \sin(x^2) \, dx \, dy \) over the triangle with vertices \( (0, 0), \ (\sqrt{\pi}, 0), \ (\sqrt{\pi}, \sqrt{\pi}) \).

   a. \( -\pi \)  
   b. \( -\sqrt{\pi} \)  
   c. 1  
   d. \( \sqrt{\pi} \)  
   e. \( \pi \)

   You must do the \( y \)-integral first because you don’t know the antiderivative of \( \sin(x^2) \).

   The edges are \( y = 0, \ x = \sqrt{\pi}, \ y = x \).

   \[
   \iint \sin(x^2) \, dx \, dy = \int_0^{\sqrt{\pi}} \int_0^x \sin(x^2) \, dy \, dx = \int_0^{\sqrt{\pi}} \left[ y \sin(x^2) \right]_{y=0}^x dx = \int_0^{\sqrt{\pi}} x \sin(x^2) \, dx = \left[ -\frac{1}{2} \cos(x^2) \right]_0^{\sqrt{\pi}} = \frac{1}{2} - \frac{1}{2} = 1
   \]
3. Find the area of the heart shaped region inside the polar curve \( r = |\theta| \).

   a. \( \frac{\pi^3}{6} \)
   b. \( \frac{\pi^3}{3} \) Correct Choice
   c. \( \frac{4\pi^3}{3} \)
   d. \( \frac{8\pi^3}{3} \)
   e. \( \frac{16\pi^3}{3} \)

   Double the upper half:
   \[
   A = 2 \int \int 1 \, dA = \int_0^\pi \int_0^\theta r \, dr \, d\theta = 2 \int_0^\pi \left[ \frac{r^2}{2} \right]_0^\theta \, d\theta = 2 \int_0^\pi \left( \frac{\theta^2}{2} \right) \, d\theta = 2 \left[ \frac{\theta^3}{6} \right]_{\theta=0}^{\pi} = \frac{\pi^3}{3}
   \]

4. Compute \( \iiint \nabla \cdot \vec{F} \, dV \) on the solid cylinder bounded by

   \( x^2 + y^2 = 9, \ z = 0 \) and \( z = 5 \) for the vector field \( \vec{F} = (x^3, y^3, z(x^2 + y^2)) \).

   a. \( 45\pi \)
   b. \( 90\pi \)
   c. \( 360\pi \)
   d. \( 810\pi \) Correct Choice
   e. \( 900\pi \)

   \( \nabla \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4x^2 + 4y^2 = 4r^2 \)

   \[
   \iiint \nabla \cdot \vec{F} \, dV = \int_0^5 \int_0^{2\pi} \int_0^3 4r^2 \, r \, dr \, d\theta \, dz = 5 \cdot 2\pi \left[ r^4 \right]_{r=0}^{3} = 810\pi
   \]
5. The solid hemisphere \( 0 \leq z \leq \sqrt{4 - x^2 - y^2} \) has density \( \delta = z \). Find the total mass.

- a. \( \pi/2 \)
- b. \( \pi \)
- c. \( 2\pi \)
- d. \( 4\pi \) Correct Choice
- e. \( 8\pi \)

In spherical coordinates, \( \delta = z = \rho \cos \varphi \) and \( J = \rho^2 \sin \varphi \).

\[
M = \iiint \rho \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \cos \varphi \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \left[ \frac{\rho^4}{4} \right]_0^2 \left[ \frac{\sin^2 \varphi}{2} \right]_{\varphi=0}^{\pi/2} = 4\pi
\]

6. The solid hemisphere \( 0 \leq z \leq \sqrt{4 - x^2 - y^2} \) has density \( \delta = z \).

Find the \( z \)-component of the center of mass.

- a. \( 1 \)
- b. \( \frac{32}{15} \pi \)
- c. \( \frac{64}{15} \pi \)
- d. \( \frac{8}{15} \pi \)
- e. \( \frac{16}{15} \) Correct Choice

\[
M_{xy} = \iiint z \rho \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \cos^2 \varphi \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = 2\pi \left[ \frac{\rho^5}{5} \right]_0^2 \left[ -\cos^3 \varphi \right]_{\varphi=0}^{\pi/2} = \frac{64}{15} \pi
\]

\[
\bar{z} = \frac{M_{xy}}{M} = \frac{64\pi}{15 \cdot 4\pi} = \frac{16}{15}
\]
7. Compute \( \int \sqrt{1 + x^2 + y^2} \, ds \) along the spiral \( \vec{r}(t) = (t \cos t, t \sin t) \) from \((0, 0)\) to \((2\pi, 0)\).

   a. \( \pi + \frac{\pi^3}{3} \)

   b. \( 2\pi + \frac{8\pi^3}{3} \) \( \text{Correct Choice} \)

   c. \( \frac{(1 + 4\pi^2)^{3/2}}{3} \)

   d. \( \frac{2(1 + 4\pi^2)^{3/2}}{3} \)

   e. \( \frac{(1 + 4\pi^2)^{3/2} - 1}{3} \)

   \( \vec{v} = (\cos t - t \sin t, \sin t + t \cos t) \)

   \[ |\vec{v}| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} = \sqrt{\cos^2 t + t^2 \cos^2 t + \sin^2 t + t^2 \sin^2 t} = \sqrt{1 + t^2} \]
   \[ \sqrt{1 + x^2 + y^2} = \sqrt{1 + t^2 \cos^2 t + t^2 \sin^2 t} = \sqrt{1 + t^2} \]
   \[ \int \sqrt{1 + x^2 + y^2} \, ds = \int_0^{2\pi} \sqrt{1 + t^2} \, |\vec{v}| \, dt = \int_0^{2\pi} \sqrt{1 + t^2} \, \frac{\sqrt{1 + t^2} \, dt}{\sqrt{1 + t^2}} = \int_0^{2\pi} (1 + t^2) dt \]
   \[ = \left[ t + \frac{t^3}{3} \right]_0^{2\pi} = 2\pi + \frac{8\pi^3}{3} \]

8. Compute \( \int y \, dx - x \, dy \) along the spiral \( \vec{r}(t) = (t \cos t, t \sin t) \) from \((0, 0)\) to \((2\pi, 0)\).

   a. \( \frac{-8\pi^3}{3} \) \( \text{Correct Choice} \)

   b. \( \frac{-4\pi^3}{3} \)

   c. \( 2\pi + \frac{8\pi^3}{3} \)

   d. \( \frac{4\pi^3}{3} \)

   e. \( \frac{8\pi^3}{3} \)

   \[ \int y \, dx - x \, dy = \int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v} \, dt \]

   where \( \vec{F} = (y, -x) = (t \sin t, -t \cos t) \) and \( \vec{v} = (\cos t - t \sin t, \sin t + t \cos t) \).

   \[ \vec{F} \cdot \vec{v} = t \sin(t \cos t - t \sin t) - t \cos(t \sin t + t \cos t) = -t^2 \sin^2 t - t^2 \cos^2 t = -t^2 \]

   \[ \int y \, dx - x \, dy = -\int_0^{2\pi} t^2 \, dt = \left. -\frac{t^3}{3} \right|_0^{2\pi} = \frac{-8\pi^3}{3} \]
Work Out: (Part credit possible. Show all work.)

9. (15 points) Compute \( \iint_R y^2 \, dx \, dy \) over the diamond shaped region \( R \) bounded by

\[ y = \frac{1}{x}, \quad y = \frac{9}{x}, \quad y = x, \quad y = 4x \]

FULL CREDIT for integrating in the curvilinear coordinates \((u, v)\) where \( u^2 = xy \) and \( v^2 = \frac{y}{x} \).

(Solve for \( x \) and \( y \).)

HALF CREDIT for integrating in rectangular coordinates.

\[
\begin{align*}
\left\{ \begin{array}{l}
\quad u^2 = xy \\
\quad v^2 = \frac{y}{x}
\end{array} \right\} & \Rightarrow \left\{ \begin{array}{l}
\quad u^2v^2 = y^2 \\
\quad \frac{u^2}{v^2} = x^2
\end{array} \right\} \Rightarrow \left\{ \begin{array}{l}
\quad x = \frac{u}{v} \\
\quad y = uv
\end{array} \right\}
\end{align*}
\]

\[
J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{array}{cc}
x & y \\
v & u
\end{array} \right| = \left| \frac{u}{v} - \frac{u}{v} \right| = 2u \frac{u}{v}
\]

\( xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1 \quad xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3 \)

\( \frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1 \quad \frac{y}{x} = 4 \Rightarrow v^2 = 4 \Rightarrow v = 2 \)

So: \( 1 \leq u \leq 3 \)

So: \( 1 \leq v \leq 2 \)

\[
\iint_R y^2 \, dx \, dy = \int_1^3 \int_1^3 u^2v^2 \ \frac{2u}{v} \ du \ dv = 2 \int_1^3 \int_1^3 u^3v \ du \ dv
\]

\[
= 2 \left[ \frac{u^4}{4} \right]_{u=1}^{u=3} \left[ \frac{v^2}{2} \right]_{v=1}^{v=3} = 2 \left[ \frac{81}{4} - \frac{1}{4} \right] \left[ \frac{4}{2} - \frac{1}{2} \right] = 60
\]
10. (20 points) Consider the hemispherical surface

\[ z = \sqrt{4 - x^2 - y^2} \]

which may be parametrized by

\[ \vec{R}(\varphi, \theta) = (2 \sin \varphi \cos \theta, 2 \sin \varphi \sin \theta, 2 \cos \varphi) \]

Find each of the following:

a. (2 pts) \( \vec{e}_\varphi = (2 \cos \varphi \cos \theta, 2 \cos \varphi \sin \theta, -2 \sin \varphi) \)

b. (2 pts) \( \vec{e}_\theta = (-2 \sin \varphi \sin \theta, 2 \sin \varphi \cos \theta, 0) \)

c. (3 pts) \( \vec{N} = \dot{\mathbf{i}} (4 \sin^2 \varphi \cos \theta) + \dot{\mathbf{j}} (-4 \sin^2 \varphi \sin \theta) + \dot{\mathbf{k}} (4 \sin \varphi \cos \varphi \cos^2 \theta + 4 \sin \varphi \cos \varphi \sin^2 \theta) \]
   \[ = (4 \sin^2 \varphi \cos \theta, 4 \sin^2 \varphi \sin \theta, 4 \sin \varphi \cos \varphi) \]

d. (2 pts) \( |\vec{N}| = \sqrt{16 \sin^4 \varphi \cos^2 \theta + 16 \sin^4 \varphi \sin^2 \theta + 16 \sin^2 \varphi \cos^2 \varphi} \]
   \[ = 4 \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} = 4 \sin \varphi \]

e. (5 pts) The total mass of the surface if the surface density is \( \delta = z \).

\[ M = \int \int \delta dS = \int \int z \ |\vec{N}| \ d\varphi \ d\theta = \int_0^{\pi/2} \int_0^{2\pi} 2 \cos \varphi \ 4 \sin \varphi \ d\varphi \ d\theta = 2\pi \cdot 8 \cdot \frac{\sin^2 \varphi}{2} \bigg|_{\varphi=0}^{\pi/2} = 8\pi \]

f. (6 pts) The \( z \)-component of the center of mass of the surface if the surface density is \( \delta = z \).

\[ M_{xy} = \int \int z \delta dS = \int \int z^2 \ |\vec{N}| \ d\varphi \ d\theta = \int_0^{\pi/2} \int_0^{2\pi} 4 \cos^2 \varphi \ 4 \sin \varphi \ d\varphi \ d\theta \]
\[ = 2\pi \cdot 16 \cdot \frac{-\cos^3 \varphi}{3} \bigg|_{\varphi=0}^{\pi/2} = \frac{32\pi}{3} \]

\[ \overline{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3 \cdot 8\pi} = \frac{4}{3} \]
11. (10 points) Consider the vector field \( \vec{F} = (-y^3, x^3, z(x^2 + y^2)) \) on the hemispherical surface of problem 10. Find each of the following:

a. (3 pts) \( \nabla \times \vec{F} = \begin{vmatrix} i & j & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y^3 & x^3 & (x^2 + y^2)z \end{vmatrix} = i(2yz) - j(2xz) + \hat{k}(3x^2 + 3y^2) = (2yz, -2xz, 3x^2 + 3y^2) \)

b. (2 pts) \( \nabla \times \vec{F}(\vec{R}(\phi, \theta)) = \)

   \( = (8 \sin \phi \cos \phi \sin \theta, -8 \sin \phi \cos \phi \cos \theta, 3 \cdot 4 \sin^2 \phi \cos^2 \theta + 3 \cdot 4 \sin^2 \phi \sin^2 \theta) \)

   \( = (8 \sin \phi \cos \phi \sin \theta, -8 \sin \phi \cos \phi \cos \theta, 12 \sin^2 \phi) \)

c. (5 pts) \( \iint \nabla \times \vec{F} \cdot d\vec{S} \) with normal pointing up.

   From problem 10, \( \vec{N} = (4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, 4 \sin \phi \cos \phi) \) which points up.

   \( \iint \nabla \times \vec{F} \cdot d\vec{S} = \iint \nabla \times \vec{F} \cdot \vec{N} d\phi d\theta \)

   \( = \iint (32 \sin^3 \phi \cos \phi \sin \theta \cos \theta - 32 \sin^3 \phi \cos \phi \sin \theta \cos \theta + 48 \sin^3 \phi \cos \phi) d\phi d\theta \)

   \( = 48 \int_0^{2\pi} \int_0^{\pi/2} \sin^3 \phi \cos \phi \sin \theta \cos \theta d\phi d\theta = 48 \cdot 2\pi \left[ \sin^4 \phi \right]_0^{\pi/2} = 24\pi \)

12. (10 points) A bowl has the shape \( z = \frac{x^2 + y^2}{3} \) for \( z \leq 3 \).

   The bowl is filled with liquid of density \( \rho = 12 - 2z \).

   Find the total mass of the liquid.

   \( M = \iiint \rho dV = \int_0^{2\pi} \int_0^3 \int_{r^2/3}^3 (12 - 2z) r \, dz \, dr \, d\theta = 2\pi \int_0^3 [12z - z_2^3]_{z=\rho/3}^3 \, r \, dz \)

   \( = 2\pi \int_0^3 [36 - 9 - \left( 4r^2 - \frac{r^4}{9} \right)] r \, dr = 2\pi \int_0^3 \left[ 27r - 4r^3 + \frac{r^5}{9} \right] dr \)

   \( = 2\pi \left[ \frac{27r^2}{2} - r^4 + \frac{r^6}{54} \right]_{r=0}^3 = 2\pi \left( \frac{243}{2} - 81 + \frac{27}{2} \right) = 108\pi \)