1. (5 points) Find the mass of the solid below \( z = x^2y \) above the region in the \( xy \)-plane between \( y = x \) and \( y = x^2 \) if the density is \( \rho(x,y,z) = 6z \).

\[
M = \iiint \rho \, dV = \int_0^1 \int_{x^2}^x \int_0^{x^2} 6z \, dy \, dz \, dx = \int_0^1 \int_0^{x^2} \left[ 3z^2 \right]_{z=0}^{x^2} \, dy \, dx = \int_0^1 3x^4y^2 \, dy \, dx \\
= \int_0^1 \left[ x^4y^3 \right]_0^x \, dx = \int_0^1 (x^7 - x^{10}) \, dx = \left[ \frac{x^8}{8} - \frac{x^{11}}{11} \right]_0^1 = \frac{1}{8} - \frac{1}{11} = \frac{11 - 8}{88} = \frac{3}{88}
\]

8. (20 points) The carbon monoxide density in the air on a certain highway is given by \( \rho = \frac{6xy^2}{z} \) where distances are measured in feet and density is measured in parts per million.

a. If a bird is at the point \((4,3,2)\) and its velocity is \((-3,4,1)\), does the bird feel the CO density increasing or decreasing and how fast?

\[
\vec{\nabla} \rho = \left( \frac{6y^2}{z}, \frac{12xy}{z}, \frac{-6xy^2}{z^2} \right) \quad \vec{\nabla} \rho \bigg|_{(4,3,2)} = (27, 72, -54) \\
d\rho \over dt = \nabla \cdot \vec{v} = (-3, 4, 1) \cdot (27, 72, -54) = -81 + 288 - 54 = 153 \quad \text{increasing}
\]

b. Use the linear approximation to estimate the CO density at the point \((3.97,3.04,2.01)\).

\[
\rho(x,y,z) \approx \rho(a,b,c) + \rho_x|_{(a,b,c)}(x-a) + \rho_y|_{(a,b,c)}(y-b) + \rho_z|_{(a,b,c)}(z-c) \\
(a,b,c) = (4,3,2) \quad (x,y,z) = (3.97,3.04,2.01) \quad (x-a,y-b,z-c) = (-.03,.04,.01) \\
\rho(a,b,c) = \rho(4,3,2) = \frac{6 \cdot 4 \cdot 3^2}{2} = 108 \quad (\rho_x,\rho_y,\rho_z)|_{(a,b,c)} = \vec{\nabla} \rho \bigg|_{(4,3,2)} = (27, 72, -54) \\
\rho(3.97,3.04,2.01) \approx 108 + 27(-.03) + 72(.04) - 54(.01) = 108 + 1.53 = 109.53
\]

c. If a bird is at the point \((4,3,2)\), in what direction should it fly to DECREASE the CO density as fast as possible?

\[
-\vec{\nabla} \rho \bigg|_{(4,3,2)} = (-27, -72, 54)
\]
9. (20 points) Find the volume of the largest rectangular box whose base is in the xy-plane, whose sides are parallel to the coordinate planes and whose top 4 vertices are on the elliptic paraboloid \[ z = 16 - 4x^2 - y^2. \]

Take \( x \) and \( y \) in the first quadrant. Then \( V = 4xyz = 4xy(16 - 4x^2 - y^2) = 64xy - 16x^3y - 4xy^3 \)
\[ V_x = 64y - 48x^2y - 4y^3 = 0 \quad V_y = 64x - 16x^3 - 12xy^2 = 0 \]
\[ V_x = 4y(16 - 12x^2 - y^2) = 0 \quad V_y = 4x(16 - 4x^2 - 3y^2) = 0 \]

If \( x \) or \( y \) is 0, then the volume is 0 and this cannot be the maximum volume.

So we solve \( 16 - 12x^2 - y^2 = 0 \) and \( 16 - 4x^2 - 3y^2 = 0 \).

Multiply the first equation by 3 and subtract the second equation:
\[ 48 - 36x^2 - 3y^2 = 0 \quad \text{minus} \quad 16 - 4x^2 - 3y^2 = 0 \quad \text{equals} \quad 32 - 32x^2 = 0 \]
So: \( x = 1 \quad y^2 = 16 - 12x^2 = 4 \quad y = 2 \)
\[ z = 16 - 4x^2 - y^2 = 16 - 4 - 4 = 8 \]
\[ V = 4xyz = 4 \cdot 1 \cdot 2 \cdot 8 = 64 \]

10. (20 points) Determine whether each of the following limits exists and say why or why not. If the limit exists, find it.

a. \( \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} \)
\[ \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2my}{x^4 + m^2x^2} = \lim_{x \to 0} \frac{mx}{x^2 + m^2} = \frac{0}{m^2} = 0 \quad \text{for all } m \text{’s.} \]
\[ \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4 + y^2} = \lim_{y \to 0} \frac{x^2mx^2}{x^4 + m^2x^2} = \lim_{y \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2} \quad \text{which is different for different } m \text{’s.} \]
So the limit does not exist.

b. \( \lim_{(x,y)\to(0,0)} \frac{y^2}{x^4 + y^2} \)
\[ \lim_{(x,y)\to(0,0)} \frac{y^2}{x^4 + y^2} = \lim_{x \to 0} \frac{m^2x^2}{x^4 + m^2x^2} = \lim_{x \to 0} \frac{m^2}{x^2 + m^2} \quad \text{which is different for different } m \text{’s.} \]
So the limit does not exist.