1. \((10 \text{ points})\) A linear map \(f : \mathbb{R}^p \to \mathbb{R}^q\) has matrix \(A = \begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 3 \end{pmatrix}\) and a linear map \(g : \mathbb{R}^q \to \mathbb{R}^p\) has matrix \(B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}\).

a. (4) What are \(p\) and \(q\)?

\(A\) is \(3 \times 2\). So \(q = 3\) and \(p = 2\).

b. (2) In the composition \(g \circ f : \mathbb{R}^p \to \mathbb{R}^n\), what is \(n\)?

In \(g \circ f\), the map \(f\) acts first. So \(n = p = 2\).

c. (4) What is the matrix of \(g \circ f\)?

\(\vec{y} = (g \circ f)(\vec{x}) = g(f(\vec{x}))\) means \(\vec{y} = g(\vec{z}) = B\vec{z}\) where \(\vec{z} = f(\vec{x}) = A\vec{x}\).

So \(\vec{y} = B\vec{z} = BA\vec{x}\). Thus the matrix of \(g \circ f\) is \(BA = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 3 \end{pmatrix}\).

2. \((20 \text{ points})\) Consider the vector space \(V = \text{Span}\{p_1, p_2, p_3, p_4\}\) where 

\[p_1 = 1 + 2x - x^3, \quad p_2 = 2 + 4x + x^4, \quad p_3 = 3 + 6x - x^3 + x^4, \quad p_4 = 2x^3 + x^4\]

Pare \(\{p_1, p_2, p_3, p_4\}\) down to a basis for \(V\). (Don’t bother proving the final set is a basis.) What is \(\text{dim} \ V\) ?

Are \(p_1, p_2, p_3, p_4\) linearly independent? Assume \(ap_1 + bp_2 + cp_3 + dp_4 = 0\).

\[a(1 + 2x - x^3) + b(2 + 4x + x^4) + c(3 + 6x - x^3 + x^4) + d(2x^3 + x^4) = 0\]

\[
\begin{align*}
2a + 4b + 6c &= 0 \\
-a - c + 2d &= 0 \\
b + c + d &= 0
\end{align*}
\]

\[
\begin{pmatrix}
2 & 4 & 6 & 0 \\
-1 & 0 & -1 & 2 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
R_2 - 2R_1 \\
R_3 + R_1 \\
\end{pmatrix}
\]

\[
\Rightarrow
\begin{pmatrix}
1 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 \\
0 & 1 & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
R_1 - 2R_4 \\
R_3 - 2R_4 \\
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
a = -s + 2t \\
b = -s - t \\
c = s \\
d = t
\end{pmatrix}
\]

\(c\) and \(d\) are free variables. So \(p_3\) and \(p_4\) can be solved for, leaving \(p_1\) and \(p_2\) as the basis.

\(\text{dim} \ V = 2\).
3. (30 points) Consider the curvilinear coordinate system \( (x, y) = \mathbf{R}(u, v) = (u v, \frac{u}{v}) \), i.e. 

\[
x = u v \quad y = \frac{u}{v}
\]

a. (5) Describe the \( u \)-coordinate curve for which \( v = 2 \).
(Give an \( xy \)-equation and describe the shape in words.)

If \( v = 2 \), then \( x = 2u, \ y = \frac{u}{2} \). So \( u = \frac{x}{2} \) and \( y = \frac{x}{4} \).
This is the line thru the origin with slope \( \frac{1}{4} \).

b. (6) Find \( \mathbf{e}_u \), the vector tangent to the \( u \)-curve at the point \( (u, v) = (1, 2) \).

\[
\mathbf{e}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = (v, \frac{1}{v}) \quad \mathbf{e}_u|_{(1,2)} = \left( 2, \frac{1}{2} \right)
\]

c. (5) Describe the \( v \)-coordinate curve for which \( u = 1 \).
(Give an \( xy \)-equation and describe the shape in words.)

If \( u = 1 \), then \( x = v, \ y = \frac{1}{v} \). So \( y = \frac{1}{x} \).
This is a hyperbola in the first and third quadrants.

d. (6) Find \( \mathbf{e}_v \), the vector tangent to the \( v \)-curve at the point \( (u, v) = (1, 2) \).

\[
\mathbf{e}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right) = \left( u, -\frac{u}{v^2} \right) \quad \mathbf{e}_v|_{(1,2)} = \left( 1, -\frac{1}{4} \right)
\]

e. (8) Let \( P \) be the pressure in a gas.

Let \( \nabla P = \left( \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right) \) be its gradient in rectangular coordinates and

let \( \nabla \left( P \circ \mathbf{R} \right) = \left( \frac{\partial \left( P \circ \mathbf{R} \right)}{\partial u}, \frac{\partial \left( P \circ \mathbf{R} \right)}{\partial v} \right) \) be its gradient in the \( u, v \)-curvilinear coordinates.

If \( \nabla P|_{(x,y)=(2,1/2)} = (16,20) \), find \( \nabla \left( P \circ \mathbf{R} \right)|_{(u,v)=(1,2)} \). HINT: Use the chain rule.

By the chain rule:

\[
\nabla \left( P \circ \mathbf{R} \right) = \nabla P \mathbf{J} \mathbf{R} = \begin{pmatrix}
\frac{\partial P}{\partial x} & \frac{\partial P}{\partial y}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix}
\]

Note: The columns of the Jacobian are the vectors \( \mathbf{e}_u \) and \( \mathbf{e}_v \).

Further, when \( (u, v) = (1, 2) \), we have \( (x, y) = (uv, \frac{u}{v}) = \left( 2, \frac{1}{2} \right) \). So

\[
\nabla \left( P \circ \mathbf{R} \right)|_{(u,v)=(1,2)} = \nabla P|_{(x,y)=(2,1/2)} \mathbf{J} \mathbf{R}|_{(u,v)=(1,2)} = (16,20) \begin{pmatrix}
2 & 1 \\
\frac{1}{2} & -\frac{1}{4}
\end{pmatrix} = (42,11)
\]
4. (40 points + 5 Extra Credit) Consider the vector spaces $V = \text{Span}\{\sinh x, \cosh x\}$ and $M(2,2) = \{2 \times 2$ matrices\}. Consider two bases on $V$:

$$\{h_1 = \sinh x, \ h_2 = \cosh x\} \quad \text{and} \quad \{e_1 = e^x, \ e_2 = e^{-x}\}$$

Consider two bases on $M(2,2)$:

$$\begin{align*}
m_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
m_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
m_3 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
m_4 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}$$

and

$$\begin{align*}
n_1 &= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \\
n_2 &= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \\
n_3 &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \\
n_4 &= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}
\end{align*}$$

Consider the linear map $L : V \rightarrow M(2,2)$ given by

$$L(f) = \begin{pmatrix} f(0) & f'(0) \\ f(\ln 2) & f'(\ln 2) \end{pmatrix}$$

Note: $e^{\ln 2} = 2, \ e^{\ln 2} = \frac{1}{2}, \ \sinh(\ln 2) = \frac{3}{4}, \ \cosh(\ln 2) = \frac{5}{4}$

a. (2) Identify the domain of $L$ and its dimension.

$\text{Dom}(L) = V \quad \text{dim} \text{Dom}(L) = 2$

b. (2) Identify the codomain of $L$ and its dimension.

$\text{Codom}(L) = M(2,2) \quad \text{dim} \text{Codom}(L) = 4$

c. (4) Is the function $L$ one-to-one? Why?

HINT: Let $f = ae^x + be^{-x}$ and $g = ce^x + de^{-x}$.

$$L(f) = \begin{pmatrix} a + b & a - b \\ 2a + \frac{b}{2} & 2a - \frac{b}{2} \end{pmatrix} \quad L(g) = \begin{pmatrix} c + d & c - d \\ 2c + \frac{d}{2} & 2c - \frac{d}{2} \end{pmatrix}$$

So $L$ is one-to-one.

d. (2 + 5 E.C.) Find the Image of $L$. Then express it as the Span of some matrices (with constant entries). What is its dimension?

$$\text{Im}(L) = \{L(f)\} = \left\{ \begin{pmatrix} a + b & a - b \\ 2a + \frac{b}{2} & 2a - \frac{b}{2} \end{pmatrix} \right\}$$

$$= \left\{ a \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} + b \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right\}$$

$\text{dim} \text{Im}(L) = 2$
e. (4) Is the function $L$ onto? Why?
   Easy Way: $L$ is not onto because $\dim \text{Codom}(L) = 4$ but $\dim \text{Im}(L) = 2$
   Hard Way: Given $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M(2, 2)$, is there an $f = ae^x + be^{-x} \in V$ such that $L(f) = M$?
   Given $p, q, r, s$, solve
   \[
   \begin{pmatrix}
   a+b & a-b \\
   2a+b/2 & 2a-b/2
   \end{pmatrix}
   =
   \begin{pmatrix}
   p & q \\
   r & s
   \end{pmatrix}
   \]
   for $a, b$.
   \[
   \begin{pmatrix}
   1 & 1 \\
   1 & -1 \\
   2 & \frac{1}{2} \\
   2 & -\frac{1}{2}
   \end{pmatrix}
   \Rightarrow 
   \begin{pmatrix}
   0 & 0 & \frac{p+q}{2} \\
   0 & 0 & \frac{p-q}{2} \\
   0 & 0 & r-\frac{3}{4}q-\frac{5}{4}p \\
   0 & 0 & s-\frac{5}{4}q-\frac{3}{4}p
   \end{pmatrix}
   \]
   No solution for general $p, q, r, s$. $L$ is not onto.

f. (4) Find the matrix of $L$ from the $h$ basis to the $m$ basis. (Call it $A_{m\rightarrow h}$)
   \[
   L(h_1) = L(\sinh x) = \begin{pmatrix} 0 & 1 \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} = m_2 + \frac{3}{4}m_3 + \frac{5}{4}m_4
   \Rightarrow 
   A_{m\rightarrow h} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
   \]
   \[
   L(h_2) = L(\cosh x) = \begin{pmatrix} 1 & 0 \\ \frac{5}{4} & \frac{3}{4} \end{pmatrix} = m_1 + \frac{5}{4}m_3 + \frac{3}{4}m_4
   \]


g. (4) Find the matrix of $L$ from the $e$ basis to the $n$ basis. (Call it $B_{n\rightarrow e}$)
   Use the definitions of $L$ and $B$, not the change of basis matrices.
   \[
   L(e_1) = L(e^x) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = n_2 + 2n_4
   \Rightarrow 
   B_{n\rightarrow e} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
   \]
   \[
   L(e_2) = L(e^{-x}) = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = n_1 + \frac{1}{2}n_3
   \]

h. (6) Find the change of basis matrices between the $e$ and $h$ bases. (Call them $C_{h\rightarrow e}$ and $C_{e\rightarrow h}$)
   Be sure to identify which is which!
   \[
   h_1 = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e_1 - \frac{1}{2}e_2
   \Rightarrow 
   C_{e\rightarrow h} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}
   \]
   \[
   h_2 = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e_1 + \frac{1}{2}e_2
   \Rightarrow 
   C_{h\rightarrow e} = C^{-1}_{e\rightarrow h} = \frac{1}{4 + \frac{1}{4}} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
   \]
i. (0) The change of basis matrices between the $m$ and $n$ bases are:

$$
C_{m-n} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -1 & 1
\end{pmatrix}
$$

and

$$
C_{n-m}^{-1} = \frac{1}{2} C_{m-n}^{-1} = \frac{1}{2}
\begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

These are given. Do not compute them!

j. (4) Recompute $B$, the matrix of $L$ from the $e$ basis to the $n$ basis by using the change of basis matrices.

$$
B_{n-e} = C_{n-m} A_{m-h} C_{h-e} = \frac{1}{2}
\begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\frac{3}{4} & \frac{5}{4} \\
\frac{5}{4} & \frac{3}{4}
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} = \ldots =
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & \frac{1}{2} \\
2 & 0
\end{pmatrix}
$$

k. (2) For the function $f = 6e^x + 4e^{-x}$, compute $L(f)$ from the definition of $L$.

$$
L(f) = \begin{pmatrix}
f(0) & f'(0) \\
f(ln2) & f'(ln2)
\end{pmatrix} = \begin{pmatrix}
10 & 2 \\
14 & 10
\end{pmatrix}
$$

l. (3) For the function $f = 6e^x + 4e^{-x}$, compute $(f)_e$ and $(f)_h$, which are the components of $f$ relative to the $e$ and $h$ bases, respectively. Check $(f)_h$ by hooking the components onto the basis.

$$
(f)_e = \begin{pmatrix}
6 \\
4
\end{pmatrix} \quad (f)_h = C_{h-e} (f)_e = \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
6 \\
4
\end{pmatrix} = \begin{pmatrix}
2 \\
10
\end{pmatrix}
$$

$$
f = 2h_1 + 10h_2 = 2\sinh x + 10\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right) + 10\left(\frac{e^x + e^{-x}}{2}\right) = 6e^x + 4e^{-x}
$$

m. (3) For the function $f = 6e^x + 4e^{-x}$, compute $[L(f)]_n$ and check by hooking the components onto the basis.

$$
[L(f)]_n = B_{n-e} (f)_e = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & \frac{1}{2} \\
2 & 0
\end{pmatrix} \begin{pmatrix}
6 \\
4
\end{pmatrix} = \begin{pmatrix}
4 \\
6 \\
2 \\
12
\end{pmatrix}
$$

$$
L(f) = 4n_1 + 6n_2 + 2n_3 + 12n_4 = 4\begin{pmatrix}
1 & -1 \\
0 & 0
\end{pmatrix} + 6\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix} + 2\begin{pmatrix}
0 & 0 \\
1 & -1
\end{pmatrix} + 12\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}
$$