MATH 304
Linear Algebra

Lecture 10:
Linear independence.
Basis of a vector space.
Linear independence

**Definition.** Let $V$ be a vector space. Vectors $v_1, v_2, \ldots, v_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0,$$

where the coefficients $r_1, \ldots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $v_1, v_2, \ldots, v_k$ are called **linearly independent**. That is, if

$$r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0 \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $v_1, \ldots, v_k \in S$. Otherwise $S$ is **linearly independent**.
Examples of linear independence

• Vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

\[ xe_1 + ye_2 + ze_3 = 0 \implies (x, y, z) = 0 \]
\[ \implies x = y = z = 0 \]

• Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

\[ aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O \]
\[ \implies a = b = c = d = 0 \]
Examples of linear independence

- Polynomials $1, x, x^2, \ldots, x^n$.

\[ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \text{ identically} \]
\[ \implies a_i = 0 \text{ for } 0 \leq i \leq n \]

- The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$.  

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x - 1)^2$.

\[ a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = a_1 + a_2(x - 1) + a_3(x - 1)^2 = \]
\[ = (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3x^2. \]

Hence $a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0$ identically
\[ \implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0 \]
\[ \implies a_1 = a_2 = a_3 = 0 \]
Problem  Let $v_1 = (1, 2, 0)$, $v_2 = (3, 1, 1)$, and $v_3 = (4, -7, 3)$. Determine whether vectors $v_2, v_2, v_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that  $r_1 v_1 + r_2 v_2 + r_3 v_3 = 0$.

This vector equation is equivalent to a system

$$
\begin{cases}
 r_1 + 3r_2 + 4r_3 = 0 \\
 2r_1 + r_2 - 7r_3 = 0 \\
 0r_1 + r_2 + 3r_3 = 0
\end{cases}
$$

The vectors $v_1, v_2, v_3$ are linearly dependent if and only if the matrix $A = (v_1, v_2, v_3)$ is singular. We obtain that $\det A = 0$. 

Theorem  The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linear combination of the other $k - 1$ vectors.

Proof:  (i) $\implies$ (ii) Suppose that
$$r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k = 0,$$
where $r_i \neq 0$ for some $1 \leq i \leq k$. Then
$$\mathbf{v}_i = -\frac{r_1}{r_i} \mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i} \mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i} \mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i} \mathbf{v}_k.$$

(ii) $\implies$ (i) Suppose that
$$\mathbf{v}_i = s_1 \mathbf{v}_1 + \cdots + s_{i-1} \mathbf{v}_{i-1} + s_{i+1} \mathbf{v}_{i+1} + \cdots + s_k \mathbf{v}_k$$
for some scalars $s_j$. Then
$$s_1 \mathbf{v}_1 + \cdots + s_{i-1} \mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1} \mathbf{v}_{i+1} + \cdots + s_k \mathbf{v}_k = 0.$$
**Theorem**  Vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n \) are linearly dependent whenever \( m > n \) (i.e., the number of coordinates is less than the number of vectors).

**Proof:** Let \( \mathbf{v}_j = (a_{1j}, a_{2j}, \ldots, a_{nj}) \) for \( j = 1, 2, \ldots, m \). Then the vector equality \( t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \cdots + t_m \mathbf{v}_m = \mathbf{0} \) is equivalent to the system

\[
\begin{align*}
  a_{11} t_1 + a_{12} t_2 + \cdots + a_{1m} t_m &= 0, \\
  a_{21} t_1 + a_{22} t_2 + \cdots + a_{2m} t_m &= 0, \\
  \quad \quad \quad \quad \quad \quad \cdots \quad \quad \quad \quad \quad \quad \cdots \\
  a_{n1} t_1 + a_{n2} t_2 + \cdots + a_{nm} t_m &= 0.
\end{align*}
\]

Note that vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) are columns of the matrix \((a_{ij})\). The number of leading entries in the row echelon form is at most \( n \). If \( m > n \) then there are free variables, therefore the zero solution is not unique.
Example. Consider vectors \( \mathbf{v}_1 = (1, -1, 1) \), \( \mathbf{v}_2 = (1, 0, 0) \), \( \mathbf{v}_3 = (1, 1, 1) \), and \( \mathbf{v}_4 = (1, 2, 4) \) in \( \mathbb{R}^3 \).

Two vectors are linearly dependent if and only if they are parallel. Hence \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent.

Vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent if and only if the matrix \( \mathbf{A} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \) is invertible.

\[
\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.
\]

Therefore \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent.

Four vectors in \( \mathbb{R}^3 \) are always linearly dependent. Thus \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) are linearly dependent.
Problem. Show that functions $e^x$, $e^{2x}$, and $e^{3x}$ are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$ae^x + be^{2x} + ce^{3x} = 0$,

$ae^x + 2be^{2x} + 3ce^{3x} = 0$,

$ae^x + 4be^{2x} + 9ce^{3x} = 0$.

It follows that $A(x)v = 0$, where

$$A(x) = \begin{pmatrix}
    e^x & e^{2x} & e^{3x} \\
    e^x & 2e^{2x} & 3e^{3x} \\
    e^x & 4e^{2x} & 9e^{3x}
\end{pmatrix}, \quad v = \begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}.$$
\[ A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \]

\[
\det A(x) = e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}
\]
\[
= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0.
\]

Since the matrix \( A(x) \) is invertible, we obtain
\[ A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0. \]
Let $f_1, f_2, \ldots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \ldots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$ 

**Theorem** If $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \ldots, f_n$ are linearly independent in $C[a, b]$. 
**Theorem 1**  Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1x}, e^{\lambda_2x}, \ldots, e^{\lambda_kx}$ are linearly independent.

**Theorem 2**  The set of functions

$$\{x^m e^{\lambda x} \mid \lambda \in \mathbb{R}, \ m = 0, 1, 2, \ldots \}$$

is linearly independent.
Spanning set

Let $S$ be a subset of a vector space $V$.

*Definition.* The span of the set $S$ is the smallest subspace $W \subset V$ that contains $S$. If $S$ is not empty then $W = \text{Span}(S)$ consists of all linear combinations $r_1v_1 + r_2v_2 + \cdots + r_kv_k$ such that $v_1, \ldots, v_k \in S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

We say that the set $S$ spans the subspace $W$ or that $S$ is a spanning set for $W$.

*Remark.* If $S_1$ is a spanning set for a vector space $V$ and $S_1 \subset S_2 \subset V$, then $S_2$ is also a spanning set for $V$. 
Basis

Definition. Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a basis.

Suppose that a set $S \subset V$ is a basis for $V$.

“Spanning set” means that any vector $v \in V$ can be represented as a linear combination

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k,$$

where $v_1, \ldots, v_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$. “Linearly independent” implies that the above representation is unique:

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k = r'_1v_1 + r'_2v_2 + \cdots + r'_kv_k$$

$$\implies (r_1 - r'_1)v_1 + (r_2 - r'_2)v_2 + \cdots + (r_k - r'_k)v_k = 0$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \ldots = r_k - r'_k = 0$$
Examples. • Standard basis for \( \mathbb{R}^n \):
\[
e_1 = (1, 0, 0, \ldots, 0, 0), \quad e_2 = (0, 1, 0, \ldots, 0, 0), \ldots, \\
e_n = (0, 0, 0, \ldots, 0, 1).
\]
Indeed, \((x_1, x_2, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n\).

• Matrices \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) form a basis for \( \mathcal{M}_{2,2}(\mathbb{R}) \).
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

• Polynomials \( 1, x, x^2, \ldots, x^{n-1} \) form a basis for \( \mathcal{P}_n = \{ a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} : a_i \in \mathbb{R} \} \).

• The infinite set \( \{ 1, x, x^2, \ldots, x^n, \ldots \} \) is a basis for \( \mathcal{P} \), the space of all polynomials.
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

**Theorem 1** If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

**Theorem 2** If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

**Theorem 3** If $k = n$ then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.
Example. Consider vectors \( \mathbf{v}_1 = (1, -1, 1), \mathbf{v}_2 = (1, 0, 0), \mathbf{v}_3 = (1, 1, 1), \) and \( \mathbf{v}_4 = (1, 2, 4) \) in \( \mathbb{R}^3 \).

Vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent (as they are not parallel), but they do not span \( \mathbb{R}^3 \).

Vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent since

\[
\begin{vmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1 \\
\end{vmatrix}
= -
\begin{vmatrix}
-1 & 1 \\
1 & 1 \\
\end{vmatrix}
= -(-2) = 2 \neq 0.
\]

Therefore \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is a basis for \( \mathbb{R}^3 \).

Vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \) span \( \mathbb{R}^3 \) (because \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) already span \( \mathbb{R}^3 \)), but they are linearly dependent.