MATH 304
Linear Algebra

Lecture 14:
Linear independence.
Spanning set

Let $S$ be a subset of a vector space $V$.

**Definition.** The span of the set $S$ is the smallest subspace $W \subset V$ that contains $S$. If $S$ is not empty then $W = \text{Span}(S)$ consists of all linear combinations $r_1v_1 + r_2v_2 + \cdots + r_kv_k$ such that $v_1, \ldots, v_k \in S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

We say that the set $S$ spans the subspace $W$ or that $S$ is a spanning set for $W$.

**Remarks.** • If $S_1$ is a spanning set for a vector space $V$ and $S_1 \subset S_2 \subset V$, then $S_2$ is also a spanning set for $V$.
• If $v_0, v_1, \ldots, v_k$ is a spanning set for $V$ and $v_0$ is a linear combination of vectors $v_1, \ldots, v_k$ then $v_1, \ldots, v_k$ is also a spanning set for $V$. 
**Linear independence**

*Definition.* Let $V$ be a vector space. Vectors $v_1, v_2, \ldots, v_k \in V$ are called **linearly dependent** if they satisfy a relation

$$ r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0, $$

where the coefficients $r_1, \ldots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $v_1, v_2, \ldots, v_k$ are called **linearly independent**. That is, if

$$ r_1 v_1 + r_2 v_2 + \cdots + r_k v_k = 0 \implies r_1 = \cdots = r_k = 0. $$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $v_1, \ldots, v_k$ in $S$. Otherwise $S$ is **linearly independent**.
Examples of linear independence

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ in $\mathbb{R}^3$.

\[ x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = 0 \implies (x, y, z) = 0 \]
\[ \implies x = y = z = 0 \]

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

\[ aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O \]
\[ \implies a = b = c = d = 0 \]
Examples of linear independence

• Polynomials $1, x, x^2, \ldots, x^n$.

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \quad \text{identically}$$

$$\implies a_i = 0 \quad \text{for} \quad 0 \leq i \leq n$$

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$.

• Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x - 1)^2$.

$$a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = a_1 + a_2(x - 1) + a_3(x - 1)^2 =$$

$$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3x^2.$$

Hence $a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = 0 \quad \text{identically}$

$$\implies a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$$

$$\implies a_1 = a_2 = a_3 = 0$$
Problem  Let \( \mathbf{v}_1 = (1, 2, 0) \), \( \mathbf{v}_2 = (3, 1, 1) \), and \( \mathbf{v}_3 = (4, -7, 3) \). Determine whether vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent.

We have to check if there exist \( r_1, r_2, r_3 \in \mathbb{R} \) not all zero such that \( r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + r_3 \mathbf{v}_3 = \mathbf{0} \).

This vector equation is equivalent to a system

\[
\begin{cases}
    r_1 + 3r_2 + 4r_3 = 0 \\
    2r_1 + r_2 - 7r_3 = 0 \\
    0r_1 + r_2 + 3r_3 = 0 \\
\end{cases}
\]

\[
\begin{pmatrix}
1 & 3 & 4 & 0 \\
2 & 1 & -7 & 0 \\
0 & 1 & 3 & 0 \\
\end{pmatrix}
\]

The vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly dependent if and only if the matrix \( A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \) is singular.

We obtain that \( \det A = 0 \).
Theorem  The following conditions are equivalent:

(i) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent;

(ii) one of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linear combination of the other $k - 1$ vectors.

Proof:  (i) $\implies$ (ii) Suppose that

$$r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\frac{r_1}{r_i} \mathbf{v}_1 - \cdots - \frac{r_{i-1}}{r_i} \mathbf{v}_{i-1} - \frac{r_{i+1}}{r_i} \mathbf{v}_{i+1} - \cdots - \frac{r_k}{r_i} \mathbf{v}_k.$$

(ii) $\implies$ (i) Suppose that

$$\mathbf{v}_i = s_1 \mathbf{v}_1 + \cdots + s_{i-1} \mathbf{v}_{i-1} + s_{i+1} \mathbf{v}_{i+1} + \cdots + s_k \mathbf{v}_k$$

for some scalars $s_j$. Then

$$s_1 \mathbf{v}_1 + \cdots + s_{i-1} \mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1} \mathbf{v}_{i+1} + \cdots + s_k \mathbf{v}_k = \mathbf{0}.$$
**Theorem**  Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

**Proof:** Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \ldots, a_{nj})$ for $j = 1, 2, \ldots, m$. Then the vector equality $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

\[
\begin{align*}
  a_{11}t_1 + a_{12}t_2 + \cdots + a_{1m}t_m &= 0, \\
  a_{21}t_1 + a_{22}t_2 + \cdots + a_{2m}t_m &= 0, \\
  &\vdots \\
  a_{n1}t_1 + a_{n2}t_2 + \cdots + a_{nm}t_m &= 0.
\end{align*}
\]

Note that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are columns of the matrix $(a_{ij})$. The number of leading entries in the row echelon form is at most $n$. If $m > n$ then there are free variables, therefore the zero solution is not unique.
Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1), \mathbf{v}_2 = (1, 0, 0), \mathbf{v}_3 = (1, 1, 1), \text{ and } \mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Two vectors are linearly dependent if and only if they are parallel. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$
\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.
$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in $\mathbb{R}^3$ are always linearly dependent. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.
Problem. Let  $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Determine whether matrices $A$, $A^2$, and $A^3$ are linearly independent.

We have $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The task is to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1A + r_2A^2 + r_3A^3 = O$.

This matrix equation is equivalent to a system

\[
\begin{align*}
-r_1 + 0r_2 + r_3 &= 0 \\
r_1 - r_2 + 0r_3 &= 0 \\
-r_1 + r_2 + 0r_3 &= 0 \\
0r_1 - r_2 + r_3 &= 0
\end{align*}
\]

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is $A + A^2 + A^3 = O$).
More facts on linear independence

Let $S_0$ and $S$ be subsets of a vector space $V$.

- If $S_0 \subset S$ and $S$ is linearly independent, then so is $S_0$.
- If $S_0 \subset S$ and $S_0$ is linearly dependent, then so is $S$.
- If $S$ is linearly independent in $V$ and $V$ is a subspace of $W$, then $S$ is linearly independent in $W$.
- The empty set is linearly independent.
- Any set containing $0$ is linearly dependent.
- Two vectors $v_1$ and $v_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $v_1$ and $v_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If $S_0$ is linearly independent and $v_0 \in V \setminus S_0$ then $S_0 \cup \{v_0\}$ is linearly independent if and only if $v_0 \not\in \text{Span}(S)$. 