MATH 304
Linear Algebra

Lecture 16:
Basis and dimension.
**Definition.** Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a **basis**.

**Theorem** A nonempty set $S \subset V$ is a basis for $V$ if and only if any vector $v \in V$ is *uniquely represented* as a linear combination

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k,$$

where $v_1, \ldots, v_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

**Remark on uniqueness.** Expansions $v = 2v_1 - v_2$, $v = -v_2 + 2v_1$, and $v = 2v_1 - v_2 + 0v_3$ are considered the same.
Examples. • Standard basis for $\mathbb{R}^n$: 
\[ e_1 = (1, 0, 0, \ldots, 0, 0), \quad e_2 = (0, 1, 0, \ldots, 0, 0), \ldots, \]
\[ e_n = (0, 0, 0, \ldots, 0, 1). \]

• Matrices 
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
form a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

• Polynomials $1, x, x^2, \ldots, x^{n-1}$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}$.

• The infinite set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis for $\mathcal{P}$, the space of all polynomials.

• The empty set is a basis for the zero vector space $\{0\}$.
Let \( \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n \) and \( r_1, r_2, \ldots, r_k \in \mathbb{R} \). The vector equation \( r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k = \mathbf{v} \) is equivalent to the matrix equation \( A \mathbf{x} = \mathbf{v} \), where

\[
A = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \quad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.
\]

That is, \( A \) is the \( n \times k \) matrix such that vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \) are consecutive columns of \( A \).

- Vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) span \( \mathbb{R}^n \) if the row echelon form of \( A \) has no zero rows.
- Vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are linearly independent if the row echelon form of \( A \) has a leading entry in each column (no free variables).
spanning linear independence

no spanning linear independence

spanning no linear independence

no spanning no linear independence
Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

**Theorem 1** If $k < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ do not span $\mathbb{R}^n$.

**Theorem 2** If $k > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

**Theorem 3** If $k = n$ then the following conditions are equivalent:

(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.
Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (as they are not parallel), but they do not span $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$
\begin{vmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 1 \\
\end{vmatrix} = -\begin{vmatrix}
-1 & 1 \\
1 & 1 \\
\end{vmatrix} = --(-2) = 2 \neq 0.
$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$ (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span $\mathbb{R}^3$), but they are linearly dependent.
Dimension

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space $V$, denoted $\dim V$, is the number of elements in any of its bases.
Examples. • \( \dim \mathbb{R}^n = n \)

• \( \mathcal{M}_{2,2}(\mathbb{R}) \): the space of \( 2 \times 2 \) matrices
  \( \dim \mathcal{M}_{2,2}(\mathbb{R}) = 4 \)

• \( \mathcal{M}_{m,n}(\mathbb{R}) \): the space of \( m \times n \) matrices
  \( \dim \mathcal{M}_{m,n}(\mathbb{R}) = mn \)

• \( \mathcal{P}_n \): polynomials of degree less than \( n \)
  \( \dim \mathcal{P}_n = n \)

• \( \mathcal{P} \): the space of all polynomials
  \( \dim \mathcal{P} = \infty \)

• \( \{0\} \): the trivial vector space
  \( \dim \{0\} = 0 \)
Problem. Find the dimension of the plane \( x + 2z = 0 \) in \( \mathbb{R}^3 \).

The general solution of the equation \( x + 2z = 0 \) is

\[
\begin{align*}
  x &= -2s \\
  y &= t \quad (t, s \in \mathbb{R}) \\
  z &= s
\end{align*}
\]

That is, \((x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1)\). Hence the plane is the span of vectors \( \mathbf{v}_1 = (0, 1, 0) \) and \( \mathbf{v}_2 = (-2, 0, 1) \). These vectors are linearly independent as they are not parallel.

Thus \( \{\mathbf{v}_1, \mathbf{v}_2\} \) is a basis so that the dimension of the plane is 2.
How to find a basis?

**Theorem** Let $S$ be a subset of a vector space $V$. Then the following conditions are equivalent:

(i) $S$ is a linearly independent spanning set for $V$, i.e., a basis;

(ii) $S$ is a minimal spanning set for $V$;

(iii) $S$ is a maximal linearly independent subset of $V$.

“Minimal spanning set” means “remove any element from this set, and it is no longer a spanning set”.

“Maximal linearly independent subset” means “add any element of $V$ to this set, and it will become linearly dependent”.
Theorem  Let $V$ be a vector space. Then

(i) any spanning set for $V$ can be reduced to a minimal spanning set;

(ii) any linearly independent subset of $V$ can be extended to a maximal linearly independent set.

Corollary 1  Any spanning set contains a basis while any linearly independent set is contained in a basis.

Corollary 2  A vector space is finite-dimensional if and only if it is spanned by a finite set.
How to find a basis?

**Approach 1.** Get a spanning set for the vector space, then reduce this set to a basis dropping one vector at a time.

**Proposition** Let \( \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k \) be a spanning set for a vector space \( V \). If \( \mathbf{v}_0 \) is a linear combination of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) then \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) is also a spanning set for \( V \).

Indeed, if \( \mathbf{v}_0 = r_1 \mathbf{v}_1 + \cdots + r_k \mathbf{v}_k \), then

\[
 t_0 \mathbf{v}_0 + t_1 \mathbf{v}_1 + \cdots + t_k \mathbf{v}_k = \\
= (t_0 r_1 + t_1) \mathbf{v}_1 + \cdots + (t_0 r_k + t_k) \mathbf{v}_k.
\]
How to find a basis?

**Approach 2.** Build a maximal linearly independent set adding one vector at a time.

If the vector space $V$ is trivial, it has the empty basis. If $V \neq \{0\}$, pick any vector $v_1 \neq 0$. If $v_1$ spans $V$, it is a basis. Otherwise pick any vector $v_2 \in V$ that is not in the span of $v_1$. If $v_1$ and $v_2$ span $V$, they constitute a basis. Otherwise pick any vector $v_3 \in V$ that is not in the span of $v_1$ and $v_2$. And so on...

**Modifications.** Instead of the empty set, we can start with any linearly independent set (if we are given one). If we are given a spanning set $S$, it is enough to pick new vectors only in $S$.

**Remark.** This inductive procedure works for finite-dimensional vector spaces. There is an analogous procedure for infinite-dimensional spaces (*transfinite induction*).
Vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$ are linearly independent.

**Problem.** Extend the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for $\mathbb{R}^3$.

Our task is to find a vector $\mathbf{v}_3$ that is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be a basis for $\mathbb{R}^3$.

**Hint 1.** $\mathbf{v}_1$ and $\mathbf{v}_2$ span the plane $x + 2z = 0$.

The vector $\mathbf{v}_3 = (1, 1, 1)$ does not lie in the plane $x + 2z = 0$, hence it is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. 
Vectors \( \mathbf{v}_1 = (0, 1, 0) \) and \( \mathbf{v}_2 = (-2, 0, 1) \) are linearly independent.

**Problem.** Extend the set \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) to a basis for \( \mathbb{R}^3 \).

Our task is to find a vector \( \mathbf{v}_3 \) that is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) will be a basis for \( \mathbb{R}^3 \).

**Hint 2.** Since vectors \( \mathbf{e}_1 = (1, 0, 0) \), \( \mathbf{e}_2 = (0, 1, 0) \), and \( \mathbf{e}_3 = (0, 0, 1) \) form a spanning set for \( \mathbb{R}^3 \), at least one of them can be chosen as \( \mathbf{v}_3 \).

Let us check that \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1 \} \) and \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3 \} \) are two bases for \( \mathbb{R}^3 \):

\[
\begin{vmatrix}
0 & -2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{vmatrix} = 1 \neq 0,
\begin{vmatrix}
0 & -2 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{vmatrix} = 2 \neq 0.
\]