Lecture 17:
Basis and dimension (continued).
Rank of a matrix.
**Definition.** Let $V$ be a vector space. A linearly independent spanning set for $V$ is called a **basis**.

Equivalently, a nonempty subset $S \subset V$ is a basis for $V$ if any vector $v \in V$ is *uniquely represented* as a linear combination

$$v = r_1v_1 + r_2v_2 + \cdots + r_kv_k,$$

where $v_1, \ldots, v_k$ are distinct vectors from $S$ and $r_1, \ldots, r_k \in \mathbb{R}$. 
Dimension

**Theorem 1** Any vector space has a basis.

**Theorem 2** If a vector space $V$ has a finite basis, then all bases for $V$ are finite and have the same number of elements.

*Definition.* The **dimension** of a vector space $V$, denoted $\dim V$, is the number of elements in any of its bases.

*Examples.*  
- $\dim \mathbb{R}^n = n$
- $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices; $\dim \mathcal{M}_{m,n} = mn$
- $\mathcal{P}_n$: polynomials of degree less than $n$; $\dim \mathcal{P}_n = n$
- $\mathcal{P}$: the space of all polynomials; $\dim \mathcal{P} = \infty$
- $\{0\}$: the trivial vector space; $\dim \{0\} = 0$
How to find a basis?

Theorem   Let $V$ be a vector space. Then

(i) any spanning set for $V$ contains a basis;
(ii) any linearly independent subset of $V$ is contained in a basis.

Approach 1. Given a spanning set for the vector space, reduce this set to a basis.

Approach 2. Given a linearly independent set, extend this set to a basis.

Approach 2a. Given a spanning set $S_1$ and a linearly independent set $S_2$, extend the set $S_2$ to a basis adding vectors from the set $S_1$. 
Vectors \( \mathbf{v}_1 = (0, 1, 0) \) and \( \mathbf{v}_2 = (-2, 0, 1) \) are linearly independent.

**Problem.** Extend the set \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) to a basis for \( \mathbb{R}^3 \).

Our task is to find a vector \( \mathbf{v}_3 \) that is not a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). Then \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) will be a basis for \( \mathbb{R}^3 \).

Since vectors \( \mathbf{e}_1 = (1, 0, 0) \), \( \mathbf{e}_2 = (0, 1, 0) \), and \( \mathbf{e}_3 = (0, 0, 1) \) form a spanning set for \( \mathbb{R}^3 \), at least one of them can be chosen as \( \mathbf{v}_3 \).

One can check that \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1 \} \) and \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3 \} \) are two bases for \( \mathbb{R}^3 \):

\[
\begin{vmatrix}
0 & -2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{vmatrix} = 1 \neq 0, \quad \begin{vmatrix}
0 & -2 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
\end{vmatrix} = 2 \neq 0.
\]
Problem. Find a basis for the vector space $V$ spanned by vectors $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$, and $w_4 = (1, 1, 1)$.

To pare this spanning set, we need to find a relation of the form $r_1 w_1 + r_2 w_2 + r_3 w_3 + r_4 w_4 = 0$, where $r_i \in \mathbb{R}$ are not all equal to zero. Equivalently,

\[
\begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 3 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
 r_1 \\
r_2 \\
r_3 \\
r_4
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

To solve this system of linear equations for $r_1, r_2, r_3, r_4$, we apply row reduction.
\[
\begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 3 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\text{ (reduced row echelon form)}
\]

\[
\begin{align*}
& r_1 + 2r_3 = 0 \\
& r_2 + r_3 = 0 \\
& r_4 = 0
\end{align*}
\iff
\[
\begin{align*}
& r_1 = -2r_3 \\
& r_2 = -r_3 \\
& r_4 = 0
\end{align*}
\]

General solution: \((r_1, r_2, r_3, r_4) = (-2t, -t, t, 0), \ t \in \mathbb{R}\).
Particular solution: \((r_1, r_2, r_3, r_4) = (2, 1, -1, 0)\).
Problem. Find a basis for the vector space $V$ spanned by vectors $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$, and $w_4 = (1, 1, 1)$.

We have obtained that $2w_1 + w_2 - w_3 = 0$. Hence any of vectors $w_1, w_2, w_3$ can be dropped. For instance, $V = \text{Span}(w_1, w_2, w_4)$.

Let us check whether vectors $w_1, w_2, w_4$ are linearly independent:

\[
\begin{vmatrix}
  1 & 0 & 1 \\
  1 & 1 & 1 \\
  0 & 1 & 1 \\
\end{vmatrix}
= \begin{vmatrix}
  1 & 0 & 1 \\
  1 & 1 & 0 \\
  0 & 1 & 0 \\
\end{vmatrix}
= \begin{vmatrix}
  1 & 1 \\
  0 & 1 \\
\end{vmatrix}
= 1 \neq 0.
\]

They are!! It follows that $V = \mathbb{R}^3$ and \{w_1, w_2, w_4\} is a basis for $V$. 
Row space of a matrix

Definition. The row space of an \( m \times n \) matrix \( A \) is the subspace of \( \mathbb{R}^n \) spanned by rows of \( A \).

The dimension of the row space is called the rank of the matrix \( A \).

Theorem 1  The rank of a matrix \( A \) is the maximal number of linearly independent rows in \( A \).

Theorem 2  Elementary row operations do not change the row space of a matrix.

Theorem 3  If a matrix \( A \) is in row echelon form, then the nonzero rows of \( A \) are linearly independent.

Corollary  The rank of a matrix is equal to the number of nonzero rows in its row echelon form.
**Theorem**  Elementary row operations do not change the row space of a matrix.

**Proof:** Suppose that $A$ and $B$ are $m \times n$ matrices such that $B$ is obtained from $A$ by an elementary row operation. Let $a_1, \ldots, a_m$ be the rows of $A$ and $b_1, \ldots, b_m$ be the rows of $B$. We have to show that $\text{Span}(a_1, \ldots, a_m) = \text{Span}(b_1, \ldots, b_m)$.

Observe that any row $b_i$ of $B$ belongs to $\text{Span}(a_1, \ldots, a_m)$. Indeed, either $b_i = a_j$ for some $1 \leq j \leq m$, or $b_i = ra_i$ for some scalar $r \neq 0$, or $b_i = a_i + ra_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\text{Span}(b_1, \ldots, b_m) \subset \text{Span}(a_1, \ldots, a_m)$.

Now the matrix $A$ can also be obtained from $B$ by an elementary row operation. By the above,

$$\text{Span}(a_1, \ldots, a_m) \subset \text{Span}(b_1, \ldots, b_m).$$
Problem. Find the rank of the matrix

\[ A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 3 & 1 \\
1 & 1 & 1
\end{pmatrix}. \]

Elementary row operations do not change the row space. Let us convert \( A \) to row echelon form:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 3 & 1 \\
1 & 1 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]
Vectors \((1, 1, 0), (0, 1, 1),\) and \((0, 0, 1)\) form a basis for the row space of \(A\). Thus the rank of \(A\) is 3.

It follows that the row space of \(A\) is the entire space \(\mathbb{R}^3\).
Problem. Find a basis for the vector space $V$ spanned by vectors $w_1 = (1, 1, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (2, 3, 1)$, and $w_4 = (1, 1, 1)$.

The vector space $V$ is the row space of a matrix

$$
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 3 & 1 \\
1 & 1 & 1
\end{pmatrix}.
$$

According to the solution of the previous problem, vectors $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ form a basis for $V$. 


Column space of a matrix

*Definition.* The **column space** of an \( m \times n \) matrix \( A \) is the subspace of \( \mathbb{R}^m \) spanned by columns of \( A \).

**Theorem 1**  The column space of a matrix \( A \) coincides with the row space of the transpose matrix \( A^T \).

**Theorem 2**  Elementary column operations do not change the column space of a matrix.

**Theorem 3**  Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

**Theorem 4**  For any matrix, the row space and the column space have the same dimension.
Problem. Find a basis for the column space of the matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
2 & 3 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\]

The column space of \( A \) coincides with the row space of \( A^T \). To find a basis, we convert \( A^T \) to row echelon form:

\[
A^T = \begin{pmatrix}
1 & 0 & 2 & 1 \\
1 & 1 & 3 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Vectors \((1, 0, 2, 1)\), \((0, 1, 1, 0)\), and \((0, 0, 0, 1)\) form a basis for the column space of \( A \).
Problem. Find a basis for the column space of the matrix

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \]

Alternative solution: We already know from a previous problem that the rank of \( A \) is 3. It follows that the columns of \( A \) are linearly independent. Therefore these columns form a basis for the column space.