Lecture 21:
Properties of linear transformations.
Range and kernel.
General linear equations.
Definition. Given vector spaces $V_1$ and $V_2$, a mapping $L : V_1 \to V_2$ is linear if

\[
\begin{align*}
L(x + y) &= L(x) + L(y), \\
L(rx) &= rL(x)
\end{align*}
\]

for any $x, y \in V_1$ and $r \in \mathbb{R}$. 
Basic properties of linear transformations

Let $L : V_1 \rightarrow V_2$ be a linear mapping.

- $L(r_1v_1 + \cdots + r_kv_k) = r_1L(v_1) + \cdots + r_kL(v_k)$ for all $k \geq 1$, $v_1, \ldots, v_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.

$L(r_1v_1 + r_2v_2) = L(r_1v_1) + L(r_2v_2) = r_1L(v_1) + r_2L(v_2),$

$L(r_1v_1 + r_2v_2 + r_3v_3) = L(r_1v_1 + r_2v_2) + L(r_3v_3) = r_1L(v_1) + r_2L(v_2) + r_3L(v_3)$, and so on.

- $L(0_1) = 0_2$, where $0_1$ and $0_2$ are zero vectors in $V_1$ and $V_2$, respectively.

$L(0_1) = L(00_1) = 0L(0_1) = 0_2.$

- $L(-v) = -L(v)$ for any $v \in V_1$.

$L(-v) = L((-1)v) = (-1)L(v) = -L(v).$
Examples of linear mappings

- **Scaling** $L : V \rightarrow V$, $L(v) = sv$, where $s \in \mathbb{R}$.

- **Dot product with a fixed vector**
  $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$, $\ell(v) = v \cdot v_0$, where $v_0 \in \mathbb{R}^n$.

- **Cross product with a fixed vector**
  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $L(v) = v \times v_0$, where $v_0 \in \mathbb{R}^3$.

- **Multiplication by a fixed matrix**
  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $L(v) = Av$, where $A$ is an $m \times n$ matrix and all vectors are column vectors.

- **Coordinate mapping**
  $L : V \rightarrow \mathbb{R}^n$, $L(v) =$ coordinates of $v$ relative to an ordered basis $v_1, v_2, \ldots, v_n$ for the vector space $V$. 
Linear mappings of functional vector spaces

- **Evaluation at a fixed point**
  \[ \ell : F(\mathbb{R}) \to \mathbb{R}, \quad \ell(f) = f(a), \text{ where } a \in \mathbb{R}. \]

- **Multiplication by a fixed function**
  \[ L : F(\mathbb{R}) \to F(\mathbb{R}), \quad L(f) = gf, \text{ where } g \in F(\mathbb{R}). \]

- **Differentiation**
  \[ D : C^1(\mathbb{R}) \to C(\mathbb{R}), \quad L(f) = f'. \]

- **Integration over a finite interval**
  \[ \ell : C(\mathbb{R}) \to \mathbb{R}, \quad \ell(f) = \int_a^b f(x) \, dx, \text{ where } a, b \in \mathbb{R}, \quad a < b. \]
More properties of linear mappings

• If a linear mapping \( L : V \rightarrow W \) is invertible then the inverse mapping \( L^{-1} : W \rightarrow V \) is also linear.

• If \( L : V \rightarrow W \) and \( M : W \rightarrow X \) are linear mappings then the composition \( M \circ L : V \rightarrow X \) is also linear.

• If \( L_1 : V \rightarrow W \) and \( L_2 : V \rightarrow W \) are linear mappings then the sum \( L_1 + L_2 \) is also linear.
Linear differential operators

- an ordinary differential operator

\[ L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2, \]

where \( g_0, g_1, g_2 \) are smooth functions on \( \mathbb{R} \).

That is, \( L(f) = g_0 f'' + g_1 f' + g_2 f \).

- Laplace’s operator \( \Delta : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2), \)

\[ \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \]

(a.k.a. the Laplacian; also denoted by \( \nabla^2 \)).
Examples. \( \mathcal{M}_{m,n}(\mathbb{R}) \): the space of \( m \times n \) matrices.

- \( \alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R}), \quad \alpha(A) = A^T \).
  \[
  \alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^T = A^T + B^T.
  \]
  \[
  \alpha(rA) = r \alpha(A) \iff (rA)^T = rA^T.
  \]
  Hence \( \alpha \) is linear.

- \( \beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}, \quad \beta(A) = \det A. \)

Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

Then \( A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

We have \( \det(A) = \det(B) = 0 \) while \( \det(A + B) = 1. \)
Hence \( \beta(A + B) \neq \beta(A) + \beta(B) \) so that \( \beta \) is not linear.
Range and kernel

Let $V$, $W$ be vector spaces and $L : V \to W$ be a linear mapping.

**Definition.** The range (or image) of $L$ is the set of all vectors $w \in W$ such that $w = L(v)$ for some $v \in V$. The range of $L$ is denoted $L(V)$.

The kernel of $L$, denoted $\ker L$, is the set of all vectors $v \in V$ such that $L(v) = 0$.

**Theorem (i)** The range of $L$ is a subspace of $W$.  
(ii) The kernel of $L$ is a subspace of $V$.  

Example. \( L : \mathbb{R}^3 \to \mathbb{R}^3, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \)

The kernel \( \ker(L) \) is the nullspace of the matrix.

\[
L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
\]

The range \( L(\mathbb{R}^3) \) is the column space of the matrix.
Example. \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \)

The range of \( L \) is spanned by vectors \((1, 1, 1), (0, 2, 0), \) and \((-1, -1, -1)\). It follows that \( L(\mathbb{R}^3) \) is the plane spanned by \((1, 1, 1)\) and \((0, 1, 0)\).

To find \( \ker(L) \), we apply row reduction to the matrix:

\[
\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Hence \((x, y, z) \in \ker(L)\) if \(x - z = y = 0\).

It follows that \( \ker(L) \) is the line spanned by \((1, 0, 1)\).
Example. $L : C^3(\mathbb{R}) \to C(\mathbb{R})$, $L(u) = u''' - 2u'' + u'$.

According to the theory of differential equations, the initial value problem

$$\begin{cases}
  u'''(x) - 2u''(x) + u'(x) = g(x), & x \in \mathbb{R}, \\
  u(a) = b_0, \\
  u'(a) = b_1, \\
  u''(a) = b_2
\end{cases}$$

has a unique solution for any $g \in C(\mathbb{R})$ and any $b_0, b_1, b_2 \in \mathbb{R}$. It follows that $L(C^3(\mathbb{R})) = C(\mathbb{R})$.

Also, the initial data evaluation $I(u) = (u(a), u'(a), u''(a))$, which is a linear mapping $I : C^3(\mathbb{R}) \to \mathbb{R}^3$, is one-to-one when restricted to $\ker(L)$. Hence $\dim \ker(L) = 3$.

It is easy to check that $L(xe^x) = L(e^x) = L(1) = 0$. Besides, the functions $xe^x$, $e^x$, and 1 are linearly independent (use Wronskian). It follows that $\ker(L) = \text{Span}(xe^x, e^x, 1)$. 
**General linear equations**

*Definition.* A **linear equation** is an equation of the form

$$L(x) = b,$$

where $L : V \rightarrow W$ is a linear mapping, $b$ is a given vector from $W$, and $x$ is an unknown vector from $V$.

The range of $L$ is the set of all vectors $b \in W$ such that the equation $L(x) = b$ has a solution.

The kernel of $L$ is the solution set of the **homogeneous** linear equation $L(x) = 0$.

*Theorem.* If the linear equation $L(x) = b$ is solvable and $\dim \ker L < \infty$, then the general solution is

$$x_0 + t_1v_1 + \cdots + t_kv_k,$$

where $x_0$ is a particular solution, $v_1, \ldots, v_k$ is a basis for the kernel of $L$, and $t_1, \ldots, t_k$ are arbitrary scalars.