Lecture 22:
General linear equations (continued).
Matrix transformations.
Matrix of a linear transformation.
Range and kernel

*Definition.* Given vector spaces $V$ and $W$, a mapping $L : V \to W$ is **linear** if

\[
L(x + y) = L(x) + L(y), \\
L(rx) = rL(x)
\]

for any $x, y \in V$ and $r \in \mathbb{R}$.

*Definition.* The **range** (or **image**) of $L$ is the set of all vectors $w \in W$ such that $w = L(v)$ for some $v \in V$. The range of $L$ is denoted $L(V)$.

The **kernel** of $L$, denoted $\ker L$, is the set of all vectors $v \in V$ such that $L(v) = 0$. 
General linear equations

Definition. A **linear equation** is an equation of the form

\[
L(x) = b,
\]

where \( L : V \rightarrow W \) is a linear mapping, \( b \) is a given vector from \( W \), and \( x \) is an unknown vector from \( V \).

The range of \( L \) is the set of all vectors \( b \in W \) such that the equation \( L(x) = b \) has a solution.

The kernel of \( L \) is the solution set of the **homogeneous** linear equation \( L(x) = 0 \).

**Theorem** If the linear equation \( L(x) = b \) is solvable and \( \dim \ker L < \infty \), then the general solution is

\[
x_0 + t_1 v_1 + \cdots + t_k v_k,
\]

where \( x_0 \) is a particular solution, \( v_1, \ldots, v_k \) is a basis for the kernel of \( L \), and \( t_1, \ldots, t_k \) are arbitrary scalars.
Example. \begin{align*}
\begin{cases}
x + y + z &= 4, \\
x + 2y &= 3.
\end{cases}
\end{align*}

\[
L : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad L \begin{pmatrix} x \\
y \\
z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\
1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\
y \\
z \end{pmatrix}.
\]

Linear equation: \( L(x) = b \), where \( b = \begin{pmatrix} 4 \\
3 \end{pmatrix} \).

\[
\begin{pmatrix} 1 & 1 & 1 & 4 \\
1 & 2 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 4 \\
0 & 1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 5 \\
0 & 1 & -1 & -1 \end{pmatrix}
\]

\[
\begin{cases}
x + 2z &= 5 \\
y - z &= -1
\end{cases} \quad \iff \quad 
\begin{cases}
x = 5 - 2z \\
y = -1 + z
\end{cases}
\]

\[
(x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1).
\]
Example. \( u''''(x) - 2u'''(x) + u''(x) = e^{2x} \).

Linear operator \( L : C^3(\mathbb{R}) \rightarrow C(\mathbb{R}) \),
\( Lu = u'''' - 2u''' + u'' \).

Linear equation: \( Lu = b \), where \( b(x) = e^{2x} \).

We know from the previous lecture that functions \( xe^x \), \( e^x \) and 1 form a basis for the kernel of \( L \). It remains to find a particular solution.

\( L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x} \).

Since \( L \) is a linear operator, \( L\left( \frac{1}{2}e^{2x} \right) = e^{2x} \).

Particular solution: \( u_0(x) = \frac{1}{2}e^{2x} \).

Thus the general solution is
\[ u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3. \]
Matrix transformations

Any $m \times n$ matrix $A$ gives rise to a transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(x) = Ax$, where $x \in \mathbb{R}^n$ and $L(x) \in \mathbb{R}^m$ are regarded as column vectors. This transformation is linear.

Example. $L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 7 \\ 0 & 5 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ be the standard basis for $\mathbb{R}^3$. We have that $L(e_1) = (1, 3, 0)$, $L(e_2) = (0, 4, 5)$, $L(e_3) = (2, 7, 8)$. Thus $L(e_1), L(e_2), L(e_3)$ are columns of the matrix.
Problem. Find a linear mapping $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that $L(e_1) = (1, 1)$, $L(e_2) = (0, -2)$, $L(e_3) = (3, 0)$, where $e_1, e_2, e_3$ is the standard basis for $\mathbb{R}^3$.

$$L(x, y, z) = L(xe_1 + ye_2 + ze_3)$$

$$= xL(e_1) + yL(e_2) + zL(e_3)$$

$$= x(1, 1) + y(0, -2) + z(3, 0) = (x + 3z, x - 2y)$$

$$L(x, y, z) = \begin{pmatrix} x + 3z \\ x - 2y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Columns of the matrix are vectors $L(e_1), L(e_2), L(e_3)$. 
**Theorem** Suppose $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix $A$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$. Columns of $A$ are vectors $L(e_1), L(e_2), \ldots, L(e_n)$, where $e_1, e_2, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$.

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = \begin{pmatrix}
    x_1 \\
x_2 \\
    \vdots \\
x_n
\end{pmatrix} \iff 
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m
\end{pmatrix} = x_1 \begin{pmatrix}
a_{11} \\
a_{21} \\
\vdots \\
a_{m1}
\end{pmatrix} + x_2 \begin{pmatrix}
a_{12} \\
a_{22} \\
\vdots \\
a_{m2}
\end{pmatrix} + \cdots + x_n \begin{pmatrix}
a_{1n} \\
a_{2n} \\
\vdots \\
a_{mn}
\end{pmatrix}
\]
Change of coordinates (revisited)

Let $V$ be a vector space.

Let $v_1, v_2, \ldots, v_n$ be a basis for $V$ and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $u_1, u_2, \ldots, u_n$ be another basis for $V$ and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself. Hence it’s represented as $x \mapsto Ux$, where $U$ is an $n \times n$ matrix. $U$ is called the transition matrix from $v_1, v_2, \ldots, v_n$ to $u_1, u_2, \ldots, u_n$. Columns of $U$ are coordinates of the vectors $v_1, v_2, \ldots, v_n$ with respect to the basis $u_1, u_2, \ldots, u_n$. 
Matrix of a linear transformation

Let $V, W$ be vector spaces and $f : V \rightarrow W$ be a linear map.

Let $v_1, v_2, \ldots, v_n$ be a basis for $V$ and $g_1 : V \rightarrow \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $w_1, w_2, \ldots, w_m$ be a basis for $W$ and $g_2 : W \rightarrow \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
g_1 \downarrow & & \downarrow g_2 \\
\mathbb{R}^n & \rightarrow & \mathbb{R}^m
\end{array}
$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence it’s represented as $x \mapsto Ax$, where $A$ is an $m \times n$ matrix. $A$ is called the **matrix of** $f$ with respect to bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$. Columns of $A$ are coordinates of vectors $f(v_1), \ldots, f(v_n)$ with respect to the basis $w_1, \ldots, w_m$. 
**Examples.**  

- $D : \mathcal{P}_3 \to \mathcal{P}_2$, $(Dp)(x) = p'(x)$.  
  Let $A_D$ be the matrix of $D$ with respect to the bases $1, x, x^2$ and $1, x$. Columns of $A_D$ are coordinates of polynomials $D1, Dx, Dx^2$ w.r.t. the basis $1, x$.  

  $D1 = 0, \quad Dx = 1, \quad Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

- $L : \mathcal{P}_3 \to \mathcal{P}_3$, $(Lp)(x) = p(x + 1)$.  
  Let $A_L$ be the matrix of $L$ w.r.t. the basis $1, x, x^2$.  

  $L1 = 1, \quad Lx = 1 + x, \quad Lx^2 = (x + 1)^2 = 1 + 2x + x^2$.  

  $\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$