MATH 304
Linear Algebra

Lecture 32:
Eigenvalues and eigenvectors of a linear operator.
**Definition.** Let $A$ be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix $A$ if $A\mathbf{v} = \lambda \mathbf{v}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$. The vector $\mathbf{v}$ is called an **eigenvector** of $A$ belonging to (or associated with) the eigenvalue $\lambda$.

If $\lambda$ is an eigenvalue of $A$ then the nullspace $\mathcal{N}(A - \lambda I)$, which is nontrivial, is called the **eigenspace** of $A$ corresponding to $\lambda$. The eigenspace consists of all eigenvectors belonging to the eigenvalue $\lambda$ plus the zero vector.
**Characteristic equation**

*Definition.* Given a square matrix $A$, the equation 
$$\det(A - \lambda I) = 0$$
is called the **characteristic equation** of $A$.

Eigenvalues $\lambda$ of $A$ are roots of the characteristic equation.

If $A$ is an $n \times n$ matrix then $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$. It is called the **characteristic polynomial** of $A$.

**Theorem** Any $n \times n$ matrix has at most $n$ eigenvalues.
**Eigenvalues and eigenvectors of an operator**

*Definition.* Let $V$ be a vector space and $L : V \to V$ be a linear operator. A number $\lambda$ is called an **eigenvalue** of the operator $L$ if $L(v) = \lambda v$ for a nonzero vector $v \in V$. The vector $v$ is called an **eigenvector** of $L$ associated with the eigenvalue $\lambda$. (If $V$ is a functional space then eigenvectors are also called **eigenfunctions**.)

If $V = \mathbb{R}^n$ then the linear operator $L$ is given by $L(x) = Ax$, where $A$ is an $n \times n$ matrix. In this case, eigenvalues and eigenvectors of the operator $L$ are precisely eigenvalues and eigenvectors of the matrix $A$. 
Suppose \( L : V \to V \) is a linear operator on a finite-dimensional vector space \( V \).

Let \( u_1, u_2, \ldots, u_n \) be a basis for \( V \) and \( g : V \to \mathbb{R}^n \) be the corresponding coordinate mapping. Let \( A \) be the matrix of \( L \) with respect to this basis. Then

\[
L(v) = \lambda v \iff A g(v) = \lambda g(v).
\]

Hence the eigenvalues of \( L \) coincide with those of the matrix \( A \). Moreover, the associated eigenvectors of \( A \) are coordinates of the eigenvectors of \( L \).

**Definition.** The characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \) of the matrix \( A \) is called the **characteristic polynomial** of the operator \( L \).

Then eigenvalues of \( L \) are roots of its characteristic polynomial.
**Theorem.** The characteristic polynomial of the operator $L$ is well defined. That is, it does not depend on the choice of a basis.

**Proof:** Let $B$ be the matrix of $L$ with respect to a different basis $v_1, v_2, \ldots, v_n$. Then $A = UBU^{-1}$, where $U$ is the transition matrix from the basis $v_1, \ldots, v_n$ to $u_1, \ldots, u_n$. We have to show that $\det(A - \lambda I) = \det(B - \lambda I)$ for all $\lambda \in \mathbb{R}$. We obtain

$$
\det(A - \lambda I) = \det(UBU^{-1} - \lambda I) \\
= \det(UBU^{-1} - U(\lambda I)U^{-1}) = \det(U(B - \lambda I)U^{-1}) \\
= \det(U) \det(B - \lambda I) \det(U^{-1}) = \det(B - \lambda I).
$$
Eigenspaces

Let \( L : V \to V \) be a linear operator. For any \( \lambda \in \mathbb{R} \), let \( V_\lambda \) denotes the set of all solutions of the equation \( L(x) = \lambda x \).

Then \( V_\lambda \) is a *subspace* of \( V \) since \( V_\lambda \) is the *kernel* of a linear operator given by \( x \mapsto L(x) - \lambda x \).

\( V_\lambda \) minus the zero vector is the set of all eigenvectors of \( L \) associated with the eigenvalue \( \lambda \).

In particular, \( \lambda \in \mathbb{R} \) is an eigenvalue of \( L \) if and only if \( V_\lambda \neq \{0\} \).

If \( V_\lambda \neq \{0\} \) then it is called the *eigenspace* of \( L \) corresponding to the eigenvalue \( \lambda \).
Example. \( V = C^\infty(\mathbb{R}), \; D : V \to V, \; Df = f'. \)

A function \( f \in C^\infty(\mathbb{R}) \) is an eigenfunction of the operator \( D \) belonging to an eigenvalue \( \lambda \) if \( f'(x) = \lambda f(x) \) for all \( x \in \mathbb{R} \).

It follows that \( f(x) = ce^{\lambda x} \), where \( c \) is a nonzero constant.

Thus each \( \lambda \in \mathbb{R} \) is an eigenvalue of \( D \).

The corresponding eigenspace is spanned by \( e^{\lambda x} \).
Example.  $V = C^\infty(\mathbb{R})$, $L : V \to V$, $Lf = f''$.

$Lf = \lambda f \iff f''(x) - \lambda f(x) = 0$ for all $x \in \mathbb{R}$.

It follows that each $\lambda \in \mathbb{R}$ is an eigenvalue of $L$ and the corresponding eigenspace $V_\lambda$ is two-dimensional.

If $\lambda > 0$ then $V_\lambda = \text{Span}(\exp(\sqrt{\lambda} x), \exp(-\sqrt{\lambda} x))$.

If $\lambda < 0$ then $V_\lambda = \text{Span}(\sin(\sqrt{-\lambda} x), \cos(\sqrt{-\lambda} x))$.

If $\lambda = 0$ then $V_\lambda = \text{Span}(1, x)$. 
Let $V$ be a vector space and $L : V \rightarrow V$ be a linear operator.

**Proposition 1** If $v \in V$ is an eigenvector of the operator $L$ then the associated eigenvalue is unique.

*Proof:* Suppose that $L(v) = \lambda_1 v$ and $L(v) = \lambda_2 v$. Then

$$
\lambda_1 v = \lambda_2 v \implies (\lambda_1 - \lambda_2)v = 0 \implies \lambda_1 - \lambda_2 = 0 \implies \lambda_1 = \lambda_2.
$$

**Proposition 2** Suppose $v_1$ and $v_2$ are eigenvectors of $L$ associated with different eigenvalues $\lambda_1$ and $\lambda_2$. Then $v_1$ and $v_2$ are linearly independent.

*Proof:* For any scalar $t \neq 0$ the vector $tv_1$ is also an eigenvector of $L$ associated with the eigenvalue $\lambda_1$. Since $\lambda_2 \neq \lambda_1$, it follows that $v_2 \neq tv_1$. That is, $v_2$ is not a scalar multiple of $v_1$. Similarly, $v_1$ is not a scalar multiple of $v_2$. 
Let $L : V \rightarrow V$ be a linear operator.

**Proposition 3** If $v_1$, $v_2$, and $v_3$ are eigenvectors of $L$ associated with distinct eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$, then they are linearly independent.

**Proof:** Suppose that $t_1 v_1 + t_2 v_2 + t_3 v_3 = 0$ for some $t_1, t_2, t_3 \in \mathbb{R}$. Then

$$L(t_1 v_1 + t_2 v_2 + t_3 v_3) = 0,$$

$$t_1 L(v_1) + t_2 L(v_2) + t_3 L(v_3) = 0,$$

$$t_1 \lambda_1 v_1 + t_2 \lambda_2 v_2 + t_3 \lambda_3 v_3 = 0.$$

It follows that

$$t_1 \lambda_1 v_1 + t_2 \lambda_2 v_2 + t_3 \lambda_3 v_3 - \lambda_3(t_1 v_1 + t_2 v_2 + t_3 v_3) = 0$$

$$\implies t_1(\lambda_1 - \lambda_3) v_1 + t_2(\lambda_2 - \lambda_3) v_2 = 0.$$  

By the above, $v_1$ and $v_2$ are linearly independent. Hence $t_1(\lambda_1 - \lambda_3) = t_2(\lambda_2 - \lambda_3) = 0 \implies t_1 = t_2 = 0$.

Then $t_3 = 0$ as well.
Theorem  If $v_1, v_2, \ldots, v_k$ are eigenvectors of a linear operator $L$ associated with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $v_1, v_2, \ldots, v_k$ are linearly independent.

Corollary 1  If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct real numbers, then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Proof:  Consider a linear operator $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ given by $Df = f'.$ Then $e^{\lambda_1 x}, \ldots, e^{\lambda_k x}$ are eigenfunctions of $D$ associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. By the theorem, the eigenfunctions are linearly independent.
Corollary 2  If \( v_1, v_2, \ldots, v_k \) are eigenvectors of a matrix \( A \) associated with distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \), then \( v_1, v_2, \ldots, v_k \) are linearly independent.

Corollary 3  Let \( A \) be an \( n \times n \) matrix such that the characteristic equation \( \det(A - \lambda I) = 0 \) has \( n \) distinct real roots. Then \( \mathbb{R}^n \) has a basis consisting of eigenvectors of \( A \).

\textit{Proof:}  Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be distinct real roots of the characteristic equation. Any \( \lambda_i \) is an eigenvalue of \( A \), hence there is an associated eigenvector \( v_i \). By Corollary 2, vectors \( v_1, v_2, \ldots, v_n \) are linearly independent. Therefore they form a basis for \( \mathbb{R}^n \).