MATH 304
Linear Algebra

Lecture 34:
Review for Test 2.
Topics for Test 2

Coordinates and linear transformations  (Leon 3.5, 4.1–4.3)
- Coordinates relative to a basis
- Change of basis, transition matrix
- Linear transformations
- Matrix transformations
- Matrix of a linear transformation

Orthogonality  (Leon 5.1–5.6)
- Inner products and norms
- Orthogonal complement, orthogonal projection
- Least squares problems
- The Gram-Schmidt orthogonalization process

Eigenvalues and eigenvectors  (Leon 6.1, 6.3)
- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization
Sample problems for Test 2

Problem 1 (15 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of $2 \times 2$ matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator $L$ with respect to the basis $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Problem 2 (20 pts.) Find a linear polynomial which is the best least squares fit to the following data:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Problem 3 (25 pts.) Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1, 1, 1, 1)$ and $x_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for $V$.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$. 
Problem 4 (30 pts.) Let \( A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \).

(i) Find all eigenvalues of the matrix \( A \).
(ii) For each eigenvalue of \( A \), find an associated eigenvector.
(iii) Is the matrix \( A \) diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix \( A^2 \).

Bonus Problem 5 (15 pts.) Let \( L : V \rightarrow W \) be a linear mapping of a finite-dimensional vector space \( V \) to a vector space \( W \). Show that

\[ \dim \text{Range}(L) + \dim \ker(L) = \dim V. \]
Problem 1. Let $M_{2,2}(\mathbb{R})$ denote the vector space of $2 \times 2$ matrices with real entries. Consider a linear operator $L : M_{2,2}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$ 

Find the matrix of the operator $L$ with respect to the basis $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $M_L$ denote the desired matrix.

By definition, $M_L$ is a $4 \times 4$ matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis $E_1, E_2, E_3, E_4$. 
\[ L(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1E_1 + 2E_2 + 0E_3 + 0E_4, \]

\[ L(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3E_1 + 4E_2 + 0E_3 + 0E_4, \]

\[ L(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0E_1 + 0E_2 + 1E_3 + 2E_4, \]

\[ L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0E_1 + 0E_2 + 3E_3 + 4E_4. \]

It follows that

\[ M_L = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix}. \]
Thus the relation

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
w_1
\end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
\]

is equivalent to the relation

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1 \\
w_1
\end{pmatrix} = \begin{pmatrix}
1 & 3 & 0 & 0 \\
2 & 4 & 0 & 0 \\
0 & 0 & 1 & 3 \\
0 & 0 & 2 & 4
\end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.
\]
Problem 2. Find a linear polynomial which is the best least squares fit to the following data:

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We are looking for a function \( f(x) = c_1 + c_2x \), where \( c_1, c_2 \) are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables \( c_1 \) and \( c_2 \):

\[
\begin{align*}
   c_1 - 2c_2 &= -3, \\
   c_1 - c_2 &= -2, \\
   c_1 &= 1, \\
   c_1 + c_2 &= 2, \\
   c_1 + 2c_2 &= 5.
\end{align*}
\]

This system is inconsistent.
We can represent the system as a matrix equation $A\mathbf{c} = \mathbf{y}$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution $\mathbf{c}$ of the above system is a solution of the normal system $A^T A \mathbf{c} = A^T \mathbf{y}$:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-2 & -1 & 0 & 1 & 2
\end{pmatrix} \begin{pmatrix}
1 & -2 \\
1 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 2
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
-2 & -1 & 0 & 1 & 2
\end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix} \\
\iff 
\begin{pmatrix}
5 & 0 \\
0 & 10
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}
$$

Thus the function $f(x) = \frac{3}{5} + 2x$ is the best least squares fit to the above data among linear polynomials.
Problem 3. Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1, 1, 1, 1)$ and $x_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for $V$.

First we apply the Gram-Schmidt orthogonalization process to vectors $x_1, x_2$ and obtain an orthogonal basis $v_1, v_2$ for the subspace $V$:

$v_1 = x_1 = (1, 1, 1, 1)$,

$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1)$.

Then we normalize vectors $v_1, v_2$ to obtain an orthonormal basis $w_1, w_2$ for $V$:

$\|v_1\| = 2 \implies w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2}(1, 1, 1, 1)$

$\|v_2\| = \sqrt{6} \implies w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)$
Problem 3. Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1, 1, 1, 1)$ and $x_2 = (1, 0, 3, 0)$.

(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Since the subspace $V$ is spanned by vectors $(1, 1, 1, 1)$ and $(1, 0, 3, 0)$, it is the row space of the matrix

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 3 & 0 \\
1 & 0 & 3 & 0
\end{pmatrix}.$$  

Then the orthogonal complement $V^\perp$ is the nullspace of $A$. To find the nullspace, we convert the matrix $A$ to reduced row echelon form:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 3 & 0 \\
1 & 0 & 3 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 3 & 0 \\
1 & 1 & 1 & 1 \\
0 & 1 & -2 & 1
\end{pmatrix}.$$
Hence a vector \((x_1, x_2, x_3, x_4) \in \mathbb{R}^4\) belongs to \(V^\perp\) if and only if
\[
\begin{pmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -2 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\]

\(\iff\) 
\[
\begin{cases}
x_1 + 3x_3 = 0 \\
x_2 - 2x_3 + x_4 = 0
\end{cases}
\iff
\begin{cases}
x_1 = -3x_3 \\
x_2 = 2x_3 - x_4
\end{cases}
\]

The general solution of the system is \((x_1, x_2, x_3, x_4) = (−3t, 2t − s, t, s) = t(−3, 2, 1, 0) + s(0, −1, 0, 1)\), where \(t, s \in \mathbb{R}\).

It follows that \(V^\perp\) is spanned by vectors \(x_3 = (0, −1, 0, 1)\) and \(x_4 = (−3, 2, 1, 0)\).
The vectors $\mathbf{x}_3 = (0, -1, 0, 1)$ and $\mathbf{x}_4 = (-3, 2, 1, 0)$ form a basis for the subspace $V^\perp$.

It remains to orthogonalize and normalize this basis:

$v_3 = x_3 = (0, -1, 0, 1)$,

$v_4 = x_4 - \frac{x_4 \cdot v_3}{v_3 \cdot v_3} v_3 = (-3, 2, 1, 0) - \frac{-2}{2} (0, -1, 0, 1)$

$= (-3, 1, 1, 1)$,

$\|v_3\| = \sqrt{2} \implies w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}} (0, -1, 0, 1)$,

$\|v_4\| = \sqrt{12} = 2\sqrt{3} \implies w_4 = \frac{v_4}{\|v_4\|} = \frac{1}{2\sqrt{3}} (-3, 1, 1, 1)$.

Thus the vectors $w_3 = \frac{1}{\sqrt{2}} (0, -1, 0, 1)$ and $w_4 = \frac{1}{2\sqrt{3}} (-3, 1, 1, 1)$ form an orthonormal basis for $V^\perp$. 
Problem 3. Let $V$ be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1, 1, 1, 1)$ and $x_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for $V$.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Alternative solution: First we extend the set $x_1, x_2$ to a basis $x_1, x_2, x_3, x_4$ for $\mathbb{R}^4$. Then we orthogonalize and normalize the latter. This yields an orthonormal basis $w_1, w_2, w_3, w_4$ for $\mathbb{R}^4$.

By construction, $w_1, w_2$ is an orthonormal basis for $V$. It follows that $w_3, w_4$ is an orthonormal basis for $V^\perp$. 
The set \( \mathbf{x}_1 = (1, 1, 1, 1), \mathbf{x}_2 = (1, 0, 3, 0) \) can be extended to a basis for \( \mathbb{R}^4 \) by adding two vectors from the standard basis.

For example, we can add vectors \( \mathbf{e}_3 = (0, 0, 1, 0) \) and \( \mathbf{e}_4 = (0, 0, 0, 1) \). To show that \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{e}_3, \mathbf{e}_4 \) is indeed a basis for \( \mathbb{R}^4 \), we check that the matrix whose rows are these vectors is nonsingular:

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{vmatrix}
= -
\begin{vmatrix}
1 & 3 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix}
= -1 \neq 0.
\]
To orthogonalize the basis \( x_1, x_2, e_3, e_4 \), we apply the Gram-Schmidt process:

\[ v_1 = x_1 = (1, 1, 1, 1), \]

\[ v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1), \]

\[ v_3 = e_3 - \frac{e_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{e_3 \cdot v_2}{v_2 \cdot v_2} v_2 = (0, 0, 1, 0) - \frac{1}{4}(1, 1, 1, 1) - \frac{2}{6}(0, -1, 2, -1) = \left( -\frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12} \right) = \frac{1}{12}(-3, 1, 1, 1), \]

\[ v_4 = e_4 - \frac{e_4 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{e_4 \cdot v_2}{v_2 \cdot v_2} v_2 - \frac{e_4 \cdot v_3}{v_3 \cdot v_3} v_3 = (0, 0, 0, 1) - \frac{1}{4}(1, 1, 1, 1) - \frac{1}{6}(0, -1, 2, -1) - \frac{1/12}{1/12} \cdot \frac{1}{12}(-3, 1, 1, 1) = \]

\[ = (0, -\frac{1}{2}, 0, \frac{1}{2}) = \frac{1}{2}(0, -1, 0, 1). \]
It remains to normalize vectors \( \mathbf{v}_1 = (1, 1, 1, 1) \), \( \mathbf{v}_2 = (0, -1, 2, -1) \), \( \mathbf{v}_3 = \frac{1}{12}(-3, 1, 1, 1) \), \( \mathbf{v}_4 = \frac{1}{2}(0, -1, 0, 1) \): 

\[
\| \mathbf{v}_1 \| = 2 \implies \mathbf{w}_1 = \frac{\mathbf{v}_1}{\| \mathbf{v}_1 \|} = \frac{1}{2}(1, 1, 1, 1)
\]

\[
\| \mathbf{v}_2 \| = \sqrt{6} \implies \mathbf{w}_2 = \frac{\mathbf{v}_2}{\| \mathbf{v}_2 \|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1)
\]

\[
\| \mathbf{v}_3 \| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} \implies \mathbf{w}_3 = \frac{\mathbf{v}_3}{\| \mathbf{v}_3 \|} = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)
\]

\[
\| \mathbf{v}_4 \| = \frac{1}{\sqrt{2}} \implies \mathbf{w}_4 = \frac{\mathbf{v}_4}{\| \mathbf{v}_4 \|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1)
\]

Thus \( \mathbf{w}_1, \mathbf{w}_2 \) is an orthonormal basis for \( V \) while \( \mathbf{w}_3, \mathbf{w}_4 \) is an orthonormal basis for \( V^\perp \).
Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix $A$.

The eigenvalues of $A$ are roots of the characteristic equation $\det(A - \lambda I) = 0$. We obtain that

$$
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{vmatrix}
$$

$$
= (1 - \lambda)^3 - 2(1 - \lambda) - 2(1 - \lambda) = (1 - \lambda)((1 - \lambda)^2 - 4)
$$

$$
= (1 - \lambda)((1 - \lambda) - 2)((1 - \lambda) + 2) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).
$$

Hence the matrix $A$ has three eigenvalues: $-1, 1, \text{ and } 3$. 

Problem 4.  Let \( A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \).

(ii) For each eigenvalue of \( A \), find an associated eigenvector.

An eigenvector \( \mathbf{v} = (x, y, z) \) of the matrix \( A \) associated with an eigenvalue \( \lambda \) is a nonzero solution of the vector equation

\[
(A - \lambda I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

To solve the equation, we convert the matrix \( A - \lambda I \) to reduced row echelon form.
First consider the case $\lambda = -1$. The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Hence

$$(A + I)v = 0 \quad \iff \quad \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $v_1 = (1, -1, 1)$ is an eigenvector of $A$ associated with the eigenvalue $-1$. 
Secondly, consider the case $\lambda = 1$. The row reduction yields

\[
A - I = \begin{pmatrix}
0 & 2 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
0 & 2 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 2 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Hence

\[(A - I)v = 0 \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}\]

The general solution is $x = -t$, $y = 0$, $z = t$, where $t \in \mathbb{R}$. In particular, $v_2 = (-1, 0, 1)$ is an eigenvector of $A$ associated with the eigenvalue 1.
Finally, consider the case \( \lambda = 3 \). The row reduction yields

\[
\begin{align*}
A - 3I &= \begin{pmatrix}
-2 & 2 & 0 \\
1 & -2 & 1 \\
0 & 2 & -2 \\
\end{pmatrix} \\
&\Rightarrow \begin{pmatrix}
1 & -1 & 0 \\
0 & 2 & -2 \\
\end{pmatrix} \\
&\Rightarrow \begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 2 & -2 \\
\end{pmatrix} \\
&\Rightarrow \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{pmatrix}.
\end{align*}
\]

Hence

\[
(A - 3I)v = 0 \iff \begin{cases}
x - z = 0, \\
y - z = 0.
\end{cases}
\]

The general solution is \( x = t, \ y = t, \ z = t \), where \( t \in \mathbb{R} \). In particular, \( v_3 = (1, 1, 1) \) is an eigenvector of \( A \) associated with the eigenvalue 3.
Problem 4. Let \( A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \).

(iii) Is the matrix \( A \) diagonalizable? Explain.

The matrix \( A \) is diagonalizable, i.e., there exists a basis for \( \mathbb{R}^3 \) formed by its eigenvectors.

Namely, the vectors \( \mathbf{v}_1 = (1, -1, 1) \), \( \mathbf{v}_2 = (-1, 0, 1) \), and \( \mathbf{v}_3 = (1, 1, 1) \) are eigenvectors of the matrix \( A \) belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) is a basis for \( \mathbb{R}^3 \).

Alternatively, the existence of a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( A \) already follows from the fact that the matrix \( A \) has three distinct eigenvalues.
Problem 4. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(iv) Find all eigenvalues of the matrix $A^2$.

Suppose that $v$ is an eigenvector of the matrix $A$ associated with an eigenvalue $\lambda$, that is, $v \neq 0$ and $Av = \lambda v$. Then

$$A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v.$$ 

Therefore $v$ is also an eigenvector of the matrix $A^2$ and the associated eigenvalue is $\lambda^2$. We already know that the matrix $A$ has eigenvalues $-1$, $1$, and $3$. It follows that $A^2$ has eigenvalues $1$ and $9$.

Since a $3 \times 3$ matrix can have up to $3$ eigenvalues, we need an additional argument to show that $1$ and $9$ are the only eigenvalues of $A^2$. One reason is that the eigenvalue $1$ has multiplicity $2$. 
**Bonus Problem 5.** Let \( L : V \to W \) be a linear mapping of a finite-dimensional vector space \( V \) to a vector space \( W \). Show that \( \dim \text{Range}(L) + \dim \ker(L) = \dim V \).

The kernel \( \ker(L) \) is a subspace of \( V \). It is finite-dimensional since the vector space \( V \) is.

Take a basis \( v_1, v_2, \ldots, v_k \) for the subspace \( \ker(L) \), then extend it to a basis \( v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m \) for the entire space \( V \).

**Claim** Vectors \( L(u_1), L(u_2), \ldots, L(u_m) \) form a basis for the range of \( L \).

Assuming the claim is proved, we obtain
\[
\dim \text{Range}(L) = m, \quad \dim \ker(L) = k, \quad \dim V = k + m.
\]
Claim  Vectors  $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Proof (spanning): Any vector $w \in \text{Range}(L)$ is represented as $w = L(v)$, where $v \in V$. Then

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m$$

for some $\alpha_i, \beta_j \in \mathbb{R}$. It follows that

$$w = L(v) = \alpha_1 L(v_1) + \cdots + \alpha_k L(v_k) + \beta_1 L(u_1) + \cdots + \beta_m L(u_m)$$

$$= \beta_1 L(u_1) + \cdots + \beta_m L(u_m).$$

Note that $L(v_i) = 0$ since $v_i \in \ker(L)$.

Thus $\text{Range}(L)$ is spanned by the vectors $L(u_1), \ldots, L(u_m)$. 
Claim  Vectors $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Proof (linear independence): Suppose that

$$t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0$$

for some $t_i \in \mathbb{R}$. Let $u = t_1 u_1 + t_2 u_2 + \cdots + t_m u_m$. Since

$$L(u) = t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0,$$

the vector $u$ belongs to the kernel of $L$. Therefore $u = s_1 v_1 + s_2 v_2 + \cdots + s_k v_k$ for some $s_j \in \mathbb{R}$. It follows that

$$t_1 u_1 + t_2 u_2 + \cdots + t_m u_m - s_1 v_1 - s_2 v_2 - \cdots - s_k v_k = u - u = 0.$$

Linear independence of vectors $v_1, \ldots, v_k, u_1, \ldots, u_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$).

Thus the vectors $L(u_1), L(u_2), \ldots, L(u_m)$ are linearly independent.