MATH 304
Linear Algebra

Lecture 40:
Review for the final exam.
Topics for the final exam: Part I

*Elementary linear algebra* (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for $2 \times 2$ and $3 \times 3$ matrices, row and column expansions, elementary row and column operations.
Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
  - Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Similarity of matrices.
Topics for the final exam: Parts III–IV

**Advanced linear algebra**  (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process
- Orthogonal polynomials
- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization
- Matrix exponentials
- Complex eigenvalues and eigenvectors
- Orthogonal matrices
- Rigid motions, rotations in space
Bases of eigenvectors

Let $A$ be an $n \times n$ matrix with real entries.

- $A$ has $n$ distinct real eigenvalues $\implies$ a basis for $\mathbb{R}^n$ formed by eigenvectors of $A$
- $A$ has complex eigenvalues $\implies$ no basis for $\mathbb{R}^n$ formed by eigenvectors of $A$
- $A$ has $n$ distinct complex eigenvalues $\implies$ a basis for $\mathbb{C}^n$ formed by eigenvectors of $A$
- $A$ has multiple eigenvalues $\implies$ further information is needed
- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of $A$ $\iff$ $A$ is symmetric: $A^T = A$
**Problem.** For each of the following $2 \times 2$ matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad (a),(b),(c): \text{yes} \]

\[ B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (a),(b),(c): \text{no} \]
Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,
(b) a basis of eigenvectors for $\mathbb{C}^2$,
(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

\[
C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \text{(a),(b): yes} \quad \text{(c): no}
\]

\[
D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{(b): yes} \quad \text{(a),(c): no}
\]
Problem. Let $V$ be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$. Consider the linear operator $D : V \rightarrow V$, $D = d/dx$.

(a) Find the matrix $A$ of the operator $D$ relative to the basis $f_1, f_2, f_3, f_4$.

(b) Find the eigenvalues of $A$.

(c) Is the matrix $A$ diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?
A is a $4 \times 4$ matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis $f_1, f_2, f_3, f_4$.

$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),

f_2'(x) = (x \cos x)' = -x \sin x + \cos x = -f_1(x) + f_4(x),

f_3'(x) = (\sin x)' = \cos x = f_4(x),

f_4'(x) = (\cos x)' = -\sin x = -f_3(x).

Thus $A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0
\end{pmatrix}$. 
Eigenvalues of $A$ are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix}
-\lambda & -1 & 0 & 0 \\
1 & -\lambda & 0 & 0 \\
1 & 0 & -\lambda & -1 \\
0 & 1 & 1 & -\lambda \\
\end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix}
0 & -\lambda & -1 \\
1 & 1 & -\lambda \\
\end{vmatrix} - (-1) \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & -\lambda \\
\end{vmatrix}$$

$$= \lambda^2(\lambda^2+1)+((\lambda^2+1)) = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2.$$

The roots are $i$ and $-i$, both of multiplicity 2.
One can show that both eigenspaces of $A$ are one-dimensional. The eigenspace for $i$ is spanned by $(0, 0, i, 1)$ and the eigenspace for $-i$ is spanned by $(0, 0, -i, 1)$. It follows that the matrix $A$ is not diagonalizable in $\mathbb{C}^4$.

There is also an indirect way to show that $A$ is not diagonalizable in $\mathbb{C}^4$. Assume the contrary. Then $A = UPU^{-1}$, where $U$ is an invertible matrix with complex entries and

$$
P = \begin{pmatrix}
  i & 0 & 0 & 0 \\
  0 & i & 0 & 0 \\
  0 & 0 & -i & 0 \\
  0 & 0 & 0 & -i
\end{pmatrix}
$$

(note that $P$ should have the same characteristic polynomial as $A$). This would imply that $A^2 = UP^2 U^{-1}$. But $P^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

Let us check if $A^2 = -I$. 
\[ A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}. \]

Since \( A^2 \neq -I \), the matrix \( A \) is not diagonalizable in \( \mathbb{C}^4 \).
Problem. Let $R$ denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $R(e_1) = e_2$, $R(e_2) = e_3$, $R(e_3) = e_1$. Is $R$ a rotation about an axis? Is $R$ a reflection in a plane?

The matrix of $R$ relative to the standard basis is

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Namely, columns of $M$ are vectors $R(e_1), R(e_2), R(e_3)$. The matrix $M$ is orthogonal since columns form an orthonormal set. Therefore $R$ is a rigid motion.

According to the classification of the $3 \times 3$ orthogonal matrices, $R$ is either a rotation about an axis, or a reflection in a plane, or the composition of a rotation about an axis with the reflection in the plane orthogonal to the axis.

We obtain that $\det M = 1$. Hence $R$ is a rotation. One can show that the angle of rotation is $120^\circ$ and the axis is the line spanned by $(1, 1, 1)$. 
Problem. Consider a linear operator \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by \( L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} \), where \( \mathbf{v}_0 = (3/5, 0, -4/5) \).

(a) Find the matrix \( B \) of the operator \( L \).
(b) Find the range and kernel of \( L \).
(c) Find the eigenvalues of \( L \).
(d) Find the matrix of the operator \( L^{2012} \) (\( L \) applied 2012 times).
\( L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5). \)

Let \( \mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3. \) Then

\[
L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
3/5 & 0 & -4/5 \\
x & y & z
\end{vmatrix}
\]

\[
= \frac{4}{5}y\mathbf{e}_1 - \left( \frac{4}{5}x + \frac{3}{5}z \right)\mathbf{e}_2 + \frac{3}{5}y\mathbf{e}_3.
\]

In particular, \( L(\mathbf{e}_1) = -\frac{4}{5}\mathbf{e}_2, \quad L(\mathbf{e}_2) = \frac{4}{5}\mathbf{e}_1 + \frac{3}{5}\mathbf{e}_3, \quad L(\mathbf{e}_3) = -\frac{3}{5}\mathbf{e}_2. \)

Therefore \( B = \begin{pmatrix}
0 & 4/5 & 0 \\
-4/5 & 0 & -3/5 \\
0 & 3/5 & 0
\end{pmatrix}. \)
The range of the operator $L$ is spanned by columns of the matrix $B$. It follows that $\text{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of $L$ is the nullspace of the matrix $B$, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

\[
\begin{pmatrix}
0 & 4/5 & 0 \\
-4/5 & 0 & -3/5 \\
0 & 3/5 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 3/4 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

$\implies x + \frac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1)$. 

\[
B = \begin{pmatrix}
0 & 4/5 & 0 \\
-4/5 & 0 & -3/5 \\
0 & 3/5 & 0 \\
\end{pmatrix}.
\]
Alternatively, the kernel of $L$ is the set of vectors $v \in \mathbb{R}^3$ such that $L(v) = v_0 \times v = 0$.

It follows that this is the line spanned by $v_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix $B$:

$$
\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}
$$

$$
= -\lambda^3 - (3/5)^2 \lambda - (4/5)^2 \lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).
$$

The eigenvalues are 0, $i$, and $-i$. 
The matrix of the operator $L^{2012}$ is $B^{2012}$.

Since the matrix $B$ has eigenvalues $0$, $i$, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = U D U^{-1}$, where $U$ is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$  

Then $B^{2012} = U D^{2012} U^{-1}$. We have that $D^{2012} = \text{diag}(0, i^{2012}, (-i)^{2012}) = \text{diag}(0, 1, 1) = -D^2$.

Hence

$$B^{2012} = U (-D^2) U^{-1} = -B^2 = \begin{pmatrix} 0.64 & 0 & 0.48 \\ 0.48 & 0 & 0.36 \end{pmatrix}. $$