Sample problems for Test 2

Any problem may be altered or replaced by a different one!

**Problem 1 (15 pts.)** Let \( \mathcal{M}_{2,2}(\mathbb{R}) \) denote the vector space of \( 2 \times 2 \) matrices with real entries. Consider a linear operator \( L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R}) \) given by

\[
L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.
\]

Find the matrix of the operator \( L \) with respect to the basis

\[
E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

**Problem 2 (20 pts.)** Find a linear polynomial which is the best least squares fit to the following data:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-3</td>
<td>-2</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

**Problem 3 (25 pts.)** Let \( V \) be a subspace of \( \mathbb{R}^4 \) spanned by the vectors \( \mathbf{x}_1 = (1,1,1,1) \) and \( \mathbf{x}_2 = (1,0,3,0) \).

(i) Find an orthonormal basis for \( V \).

(ii) Find an orthonormal basis for the orthogonal complement \( V^\perp \).

**Problem 4 (30 pts.)** Let \( A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \).

(i) Find all eigenvalues of the matrix \( A \).

(ii) For each eigenvalue of \( A \), find an associated eigenvector.

(iii) Is the matrix \( A \) diagonalizable? Explain.

(iv) Find all eigenvalues of the matrix \( A^2 \).

**Bonus Problem 5 (15 pts.)** Let \( L : V \to W \) be a linear mapping of a finite-dimensional vector space \( V \) to a vector space \( W \). Show that

\[
\dim \text{Range}(L) + \dim \ker(L) = \dim V.
\]