Sample problems for the final exam
Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial \( p(x) \) such that \( p(-1) = p(3) = 6 \) and \( p'(2) = p(1) \).

Problem 2 (20 pts.) Let \( A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix} \).

(i) Evaluate the determinant of the matrix \( A \).
(ii) Find the inverse matrix \( A^{-1} \).

Problem 3 (20 pts.) Let \( v_1 = (1, 1, 1), v_2 = (1, 1, 0), \) and \( v_3 = (1, 0, 1) \). Let \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be a linear operator on \( \mathbb{R}^3 \) such that \( L(v_1) = v_2, L(v_2) = v_3, L(v_3) = v_1 \).

(i) Show that the vectors \( v_1, v_2, v_3 \) form a basis for \( \mathbb{R}^3 \).
(ii) Find the matrix of the operator \( L \) relative to the basis \( v_1, v_2, v_3 \).
(iii) Find the matrix of the operator \( L \) relative to the standard basis.

Problem 4 (25 pts.) Let \( B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \).

(i) Find all eigenvalues of the matrix \( B \).
(ii) Find a basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( B \).
(iii) Find an orthonormal basis for \( \mathbb{R}^3 \) consisting of eigenvectors of \( B \).
(iv) Find a diagonal matrix \( D \) and an invertible matrix \( U \) such that \( B = UDU^{-1} \).

Problem 5 (20 pts.) Let \( V \) be a subspace of \( \mathbb{R}^4 \) spanned by vectors \( x_1 = (1, 1, 0, 0), x_2 = (2, 0, -1, 1), \) and \( x_3 = (0, 1, 1, 0) \).

(i) Find the distance from the point \( y = (0, 0, 0, 4) \) to the subspace \( V \).
(ii) Find the distance from the point \( y \) to the orthogonal complement \( V^\perp \).
**Bonus Problem 6 (15 pts.)** Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(x) = Cx,$$

where

$$C = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$ 

The operator $K$ is a rotation about an axis.

(i) Find the axis of rotation.

(ii) Find the angle of rotation.

**Bonus Problem 7 (15 pts.)** Let $P$ be a square matrix. Assuming $P$ is diagonalizable, prove that $\det(\exp P) = e^{\text{trace}(P)}$. 