Lecture 2:
Gaussian elimination (continued).
Row echelon form.
Gauss-Jordan reduction.
System of linear equations

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  & \quad \cdots \cdots \cdots \cdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Here \( x_1, x_2, \ldots, x_n \) are variables and \( a_{ij}, b_j \) are constants.

A solution of the system is a common solution of all equations in the system.

A system of linear equations can have one solution, infinitely many solutions, or no solution at all.
\[
\begin{align*}
    x - y &= -2 \\
    2x + 3y &= 6 \\
    x &= 0, \ y &= 2
\end{align*}
\]
\begin{align*}
2x + 3y &= 2 \\
2x + 3y &= 6
\end{align*}

\textit{inconsistent system} (no solutions)
\[
\begin{align*}
\begin{cases}
4x + 6y &= 12 \\
2x + 3y &= 6
\end{cases}
\iff 2x + 3y = 6
\end{align*}
\]
Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

\[
x - y = 2 \quad \implies \quad x = y + 2
\]

\[
2x - y - z = 5 \quad \implies \quad 2(y + 2) - y - z = 5
\]

After the elimination is completed, the system is solved by back substitution.

\[
y = 1 \quad \implies \quad x = y + 2 = 3
\]
Gaussian elimination

*Gaussian elimination* is a modification of the elimination method that allows only so-called *elementary operations*.

*Elementary operations* for systems of linear equations:
(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another equation;
(3) to interchange two equations.

**Theorem (i)** Applying elementary operations to a system of linear equations does not change the solution set of the system.  **(ii)** Any elementary operation can be undone by another elementary operation.
**Operation 1:** multiply the $i$th equation by $r \neq 0$. 

\[
\begin{align*}
\begin{cases}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  \cdots & \cdots \\
  a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\
  \cdots & \cdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\
\end{cases}
\end{align*}
\]

\[
\implies \begin{cases}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  \cdots & \cdots \\
  (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n &= rb_i \\
  \cdots & \cdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\
\end{cases}
\]

To undo the operation, multiply the $i$th equation by $r^{-1}$. 
Operation 2: add \( r \) times the \( i \)th equation to the \( j \)th equation.

\[
\begin{align*}
\cdots \cdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i \\
\cdots \cdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j \\
\cdots \cdots 
\end{align*}
\Rightarrow
\]

\[
\begin{align*}
\cdots \cdots \\
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
\cdots \cdots \\
(a_{j1} + ra_{i1})x_1 + \cdots + (a_{jn} + ra_{in})x_n &= b_j + rb_i \\
\cdots \cdots 
\end{align*}
\]

To undo the operation, add \(-r\) times the \( i \)th equation to the \( j \)th equation.
Operation 3: interchange the $i$th and $j$th equations.

\[
\begin{cases}
\ldots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\ldots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
\ldots
\end{cases}
\]

\[\Rightarrow\]

\[
\begin{cases}
\ldots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j \\
\ldots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\ldots
\end{cases}
\]

To undo the operation, apply it once more.
Solution of a system of linear equations splits into two parts: (A) elimination and (B) back substitution. Both parts can be done by applying a finite number of elementary operations.

Example.

\[
\begin{align*}
\begin{cases}
  x - y &= 2 \\
  2x - y - z &= 3 \\
  x + y + z &= 6
  \end{cases}
  \rightarrow
\begin{cases}
  x - y &= 2 \\
  y - z &= -1 \\
  2y + z &= 4
  \end{cases}
  \rightarrow
\begin{cases}
  x &= 3 \\
  y &= 1 \\
  z &= 2
  \end{cases}
\end{align*}
\]
Another example.

\[
\begin{aligned}
x + y - 2z &= 1 \\
y - z &= 3 \\
-x + 4y - 3z &= 14
\end{aligned}
\]

Add the 1st equation to the 3rd equation:

\[
\begin{aligned}
x + y - 2z &= 1 \\
y - z &= 3 \\
5y - 5z &= 15
\end{aligned}
\]

Add \(-5\) times the 2nd equation to the 3rd equation:

\[
\begin{aligned}
x + y - 2z &= 1 \\
y - z &= 3 \\
0 &= 0
\end{aligned}
\]
Add $-1$ times the 2nd equation to the 1st equation:

\[
\begin{aligned}
    x - z &= -2 \\
y - z &= 3 \\
0 &= 0
\end{aligned}
\]

\[
\begin{aligned}
x &= z - 2 \\
y &= z + 3
\end{aligned}
\]

Here $z$ is a free variable (\(x\) and \(y\) are leading variables).

It follows that

\[
\begin{aligned}
x &= t - 2 \\
y &= t + 3 \\
z &= t
\end{aligned}
\]

for some \(t \in \mathbb{R}\).
System of linear equations:
\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 14
\end{align*}
\]

Solution: \((x, y, z) = (t - 2, t + 3, t), \quad t \in \mathbb{R}\).

In vector form, \((x, y, z) = (-2, 3, 0) + t(1, 1, 1)\).
Yet another example.

\[
\begin{aligned}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 1
\end{aligned}
\]

Add the 1st equation to the 3rd equation:

\[
\begin{aligned}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  5y - 5z &= 2
\end{aligned}
\]

Add \(-5\) times the 2nd equation to the 3rd equation:

\[
\begin{aligned}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  0 &= -13
\end{aligned}
\]
System of linear equations:
\[
\begin{align*}
    x + y - 2z &= 1 \\
    y - z &= 3 \\
    -x + 4y - 3z &= 1
\end{align*}
\]

Solution: no solution (inconsistent system).
Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples:

\[
\begin{pmatrix}
2 & 7 \\
-1 & 0 \\
3 & 3
\end{pmatrix},
\begin{pmatrix}
2 & 7 & 0.2 \\
4.6 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
\frac{3}{5} \\
\frac{5}{8} \\
4
\end{pmatrix},
(\sqrt{2}, 0, -\sqrt{3}, 5),
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

Dimensions = (# of rows) \times (# of columns)

n-by-n: *square matrix*

n-by-1: *column vector*

1-by-n: *row vector*
System of linear equations:

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \vdots & \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Coefficient matrix and column vector of the right-hand sides:

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{pmatrix}
\]
System of linear equations:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    & \cdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Augmented matrix:

\[
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{pmatrix}
\]
Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the augmented matrix.

Elementary operations for systems of linear equations correspond to **elementary row operations** for augmented matrices:

1. to multiply a row by a nonzero scalar;
2. to add the $i$th row multiplied by some $r \in \mathbb{R}$ to the $j$th row;
3. to interchange two rows.

**Remark.** Rows are added and multiplied by scalars as vectors (namely, row vectors).
**Elementary row operations**

Augmented matrix:

\[
\begin{pmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\
  a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \\
\end{pmatrix}
= \begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_m
\end{pmatrix},
\]

where \( v_i = (a_{i1} \ a_{i2} \ \ldots \ a_{in} \mid b_i) \) is a row vector.
**Operation 1:** to multiply the $i$th row by $r \neq 0$:

\[
\begin{pmatrix}
v_1 \\
\vdots \\
v_i \\
\vdots \\
v_m
\end{pmatrix}
\rightarrow
\begin{pmatrix}
v_1 \\
\vdots \\
r v_i \\
\vdots \\
v_m
\end{pmatrix}
\]
Elementary row operations

**Operation 2:** to add the $i$th row multiplied by $r$ to the $j$th row:

\[
\begin{pmatrix}
v_1 \\
\vdots \\
v_i \\
v_j \\
\vdots \\
v_m
\end{pmatrix}
\rightarrow
\begin{pmatrix}
v_1 \\
\vdots \\
v_i \\
v_j + r v_i \\
\vdots \\
v_m
\end{pmatrix}
\]
Elementary row operations

Operation 3: to interchange the $i$th row with the $j$th row:

$$
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_i \\
  \vdots \\
  v_j \\
  \vdots \\
  v_m
\end{pmatrix} \rightarrow
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_j \\
  \vdots \\
  v_i \\
  \vdots \\
  v_m
\end{pmatrix}
$$
Row echelon form

**Definition.** Leading entry of a matrix is the first nonzero entry in a row.

The goal of the Gaussian elimination is to convert the augmented matrix into row echelon form:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

\[
\begin{pmatrix}
1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\
0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Row echelon form

General augmented matrix in row echelon form:

- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.
Strict triangular form is a particular case of row echelon form that can occur for systems of $n$ equations in $n$ variables:

- no zero rows;
- no free variables.
Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

Augmented matrix of an inconsistent system
The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

\[
\begin{bmatrix}
1 & * & * & * & * & * & * \\
1 & * & * & * & * & * & * \\
1 & * & * & * & * & * & * \\
1 & * & * & * & * & * & * \\
1 & * & * & * & * & * & * \\
1 & * & * & * & * & * & * \\
\end{bmatrix}
\]

- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.
Example.

\[
\begin{align*}
  x - y &= 2 \\
  2x - y - z &= 3 \\
  x + y + z &= 6
\end{align*}
\]

Row echelon form (also strict triangular):

\[
\begin{align*}
  x - y &= 2 \\
  y - z &= -1 \\
  z &= 2
\end{align*}
\]

Reduced row echelon form:

\[
\begin{align*}
  x &= 3 \\
  y &= 1 \\
  z &= 2
\end{align*}
\]
Another example.

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  -x + 4y - 3z &= 14
\end{align*}
\]

Row echelon form:

\[
\begin{align*}
  x + y - 2z &= 1 \\
  y - z &= 3 \\
  0 &= 0
\end{align*}
\]

Reduced row echelon form:

\[
\begin{align*}
  x - z &= -2 \\
  y - z &= 3 \\
  0 &= 0
\end{align*}
\]
Yet another example.

\[
\begin{align*}
\begin{cases}
x + y - 2z &= 1 \\
y - z &= 3 \\
-x + 4y - 3z &= 1
\end{cases}
\end{align*}
\]

Row echelon form:

\[
\begin{align*}
\begin{cases}
x + y - 2z &= 1 \\
y - z &= 3 \\
0 &= 1
\end{cases}
\end{align*}
\]

Reduced row echelon form:

\[
\begin{align*}
\begin{cases}
x - z &= 0 \\
y - z &= 0 \\
0 &= 1
\end{cases}
\end{align*}
\]
How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.
New example. \[
\begin{align*}
\begin{cases}
  x_2 + 2x_3 + 3x_4 &= 6 \\
  x_1 + 2x_2 + 3x_3 + 4x_4 &= 10
\end{cases}
\end{align*}
\]

Variables: \(x_1, x_2, x_3, x_4\).

Augmented matrix: \[
\begin{pmatrix}
 0 & 1 & 2 & 3 & | & 6 \\
 1 & 2 & 3 & 4 & | & 10
\end{pmatrix}
\]

To get it into row echelon form, we exchange the two rows:

\[
\begin{pmatrix}
 1 & 2 & 3 & 4 & | & 10 \\
 0 & 1 & 2 & 3 & | & 6
\end{pmatrix}
\]

Consistency check is passed. To convert into reduced row echelon form, add \(-2\) times the 2nd row to the 1st row:

\[
\begin{pmatrix}
 1 & 0 & -1 & -2 & | & -2 \\
 0 & 1 & 2 & 3 & | & 6
\end{pmatrix}
\]

The leading variables are \(x_1\) and \(x_2\); hence \(x_3\) and \(x_4\) are free variables.
Back to the system:
\[
\begin{align*}
  x_1 - x_3 - 2x_4 &= -2 \\
  x_2 + 2x_3 + 3x_4 &= 6
\end{align*}
\]
\[
\iff
\begin{align*}
  x_1 &= x_3 + 2x_4 - 2 \\
  x_2 &= -2x_3 - 3x_4 + 6
\end{align*}
\]

General solution:
\[
\begin{align*}
  x_1 &= t + 2s - 2 \\
  x_2 &= -2t - 3s + 6 \\
  x_3 &= t \\
  x_4 &= s
\end{align*}
\]
\((t, s \in \mathbb{R})\)

In vector form, \((x_1, x_2, x_3, x_4) = \)
\[
= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).
\]
Example with a parameter.

\[
\begin{cases}
y + 3z = 0 \\
x + y - 2z = 0 \quad (a \in \mathbb{R}) \\
x + 2y + az = 0
\end{cases}
\]

The system is **homogeneous** (all right-hand sides are zeros). Therefore it is consistent \((x = y = z = 0\) is a solution).

Augmented matrix:

\[
\begin{pmatrix}
0 & 1 & 3 & 0 \\
1 & 1 & -2 & 0 \\
1 & 2 & a & 0
\end{pmatrix}
\]

Since the 1st row cannot serve as a pivotal one, we interchange it with the 2nd row:
\[
\begin{pmatrix}
0 & 1 & 3 & | & 0 \\
1 & 1 & -2 & | & 0 \\
1 & 2 & a & | & 0
\end{pmatrix}
\rightarrow 
\begin{pmatrix}
1 & 1 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
0 & 1 & a+2 & | & 0
\end{pmatrix}
\]

Now we can start the elimination.

First subtract the 1st row from the 3rd row:

\[
\begin{pmatrix}
1 & 1 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
1 & 2 & a & | & 0
\end{pmatrix}
\rightarrow 
\begin{pmatrix}
1 & 1 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
0 & 1 & a+2 & | & 0
\end{pmatrix}
\]

The 2nd row is our new pivotal row.

Subtract the 2nd row from the 3rd row:

\[
\begin{pmatrix}
1 & 1 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
0 & 1 & a+2 & | & 0
\end{pmatrix}
\rightarrow 
\begin{pmatrix}
1 & 1 & -2 & | & 0 \\
0 & 1 & 3 & | & 0 \\
0 & 0 & a-1 & | & 0
\end{pmatrix}
\]
At this point row reduction splits into two cases.

**Case 1:** $a \neq 1$. In this case, multiply the 3rd row by $(a - 1)^{-1}$:

$$
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & a - 1 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

The matrix is converted into row echelon form. We proceed towards reduced row echelon form.

Subtract 3 times the 3rd row from the 2nd row:

$$
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$
Add 2 times the 3rd row to the 1st row:
\[
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Finally, subtract the 2nd row from the 1st row:
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Thus \( x = y = z = 0 \) is the only solution.
Case 2: $a = 1$. In this case, the matrix is already in row echelon form:

$$
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

To get reduced row echelon form, subtract the 2nd row from the 1st row:

$$
\begin{pmatrix}
1 & 1 & -2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & -5 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

$z$ is a free variable.

$$
\begin{cases}
x - 5z = 0 \\
y + 3z = 0
\end{cases} \quad \iff \quad 
\begin{cases}
x = 5z \\
y = -3z
\end{cases}
$$
System of linear equations:

\[
\begin{cases}
  y + 3z = 0 \\
  x + y - 2z = 0 \\
  x + 2y + az = 0
\end{cases}
\]

Solution: If \( a \neq 1 \) then \((x, y, z) = (0, 0, 0)\); if \( a = 1 \) then \((x, y, z) = (5t, -3t, t), \ t \in \mathbb{R}\).