MATH 409
Advanced Calculus I

Lecture 8:
Monotone sequences (continued).
Cauchy sequences.
Limit points.
Monotone sequences

**Definition.** A sequence \( \{x_n\} \) is called increasing (or nondecreasing) if \( x_n \leq x_{n+1} \) for all \( n \in \mathbb{N} \). It is called strictly increasing if \( x_n < x_{n+1} \) for all \( n \in \mathbb{N} \).

Likewise, the sequence \( \{x_n\} \) is called decreasing (or nonincreasing) if \( x_n \geq x_{n+1} \) for all \( n \in \mathbb{N} \). It is strictly decreasing if \( x_n > x_{n+1} \) for all \( n \in \mathbb{N} \).

Increasing and decreasing sequences are called monotone.

**Theorem** Any monotone sequence converges to a limit if bounded, and diverges to infinity otherwise.
**Examples**

- **If** \(0 < a < 1\) **then** \(a^n \to 0\) **as** \(n \to \infty\).

  Since \(a < 1\) and \(a > 0\), it follows that \(a^{n+1} < a^n\) and \(a^n > 0\) for all \(n \in \mathbb{N}\). Hence the sequence \(\{a^n\}\) is strictly decreasing and bounded. Therefore it converges to some \(x \in \mathbb{R}\). Since \(a^{n+1} = a^n a\) for all \(n\), it follows that \(a^{n+1} \to xa\) as \(n \to \infty\). However the sequence \(\{a^{n+1}\}\) is a subsequence of \(\{a^n\}\), hence it converges to the same limit as \(\{a^n\}\). Thus \(xa = x\), which implies that \(x = 0\).

- **If** \(a > 1\) **then** \(a^n \to +\infty\) **as** \(n \to \infty\).

  Since \(a > 1\), it follows that \(a^{n+1} > a^n > 1\) for all \(n \in \mathbb{N}\). Hence the sequence \(\{a^n\}\) is strictly increasing. Then \(\{a^n\}\) either diverges to \(+\infty\) or converges to a limit \(x\). In the latter case we argue as above to obtain that \(x = 0\). However this contradicts with \(a^n > 1\). Thus \(\{a^n\}\) diverges to \(+\infty\).
Examples

- If $a > 0$ then $\sqrt[n]{a} \to 1$ as $n \to \infty$.

Remark. By definition, $\sqrt[n]{a}$ is a unique positive number $r$ such that $r^n = a$.

If $a \geq 1$ then $a^{n+1} \geq a^n \geq 1$ for all $n \in \mathbb{N}$, which implies that $\sqrt[n]{a^{n+1}} \geq \sqrt[n]{a^n} \geq 1$. Notice that $\sqrt[n]{a^{n+1}} = \sqrt[n]{a}$ and $\sqrt[n]{a^n} = \sqrt[n+1]{a}$. Hence $\sqrt[n]{a} \geq \sqrt[n+1]{a} \geq 1$ for all $n$.

Similarly, in the case $0 < a < 1$ we obtain that $\sqrt[n]{a} < \sqrt[n+1]{a} < 1$ for all $n$.

In either case, the sequence $\{\sqrt[n]{a}\}$ is monotone and bounded. Therefore it converges to a limit $x$. Then the sequence $\{\sqrt[2n]{a}\}$ also converges to $x$ since it is a subsequence of $\{\sqrt[n]{a}\}$. At the same time, $(\sqrt[2n]{a})^2 = \sqrt[n]{a}$, which implies that $x^2 = x$. Hence $x = 0$ or $x = 1$. However the limit cannot be 0 since $\sqrt[n]{a} \geq \min(a, 1) > 0$. Thus $x = 1$. 
Examples

- The sequence \( x_n = \left(1 + \frac{1}{n}\right)^n \), \( n = 1, 2, 3, \ldots \), is increasing and bounded, hence it is convergent.

Remark. The limit is the number \( e = 2.71828 \ldots \)

First let us show that \( \{x_n\} \) is increasing. For any \( n \in \mathbb{N} \),

\[
x_n = \left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n^n}.
\]

If \( n \geq 2 \) then, similarly, \( x_{n-1} = \frac{n^{n-1}}{(n-1)^{n-1}} \). Hence

\[
\frac{x_n}{x_{n-1}} = \frac{(n+1)^n}{n^n} \cdot \frac{(n-1)^{n-1}}{n^{n-1}} = \left(\frac{(n+1)(n-1)}{n^2}\right)^{n-1} \cdot \frac{n+1}{n} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right).
\]
To proceed, we need the following estimate.

**Lemma** If $0 < x < 1$, then $(1 - x)^k \geq 1 - kx$ for all $k \in \mathbb{N}$.

Using the lemma, we obtain that

$$\frac{x_n}{x_{n-1}} = \left(1 - \frac{1}{n^2}\right)^{n-1} \left(1 + \frac{1}{n}\right) \geq \left(1 - \frac{n-1}{n^2}\right) \left(1 + \frac{1}{n}\right)$$

$$= 1 - \frac{n-1}{n^2} + \frac{1}{n} - \frac{n-1}{n^3} = 1 + \frac{1}{n^2} - \frac{n-1}{n^3} = 1 + \frac{1}{n^3} > 1.$$

Thus the sequence $\{x_n\}$ is strictly increasing.

**Proof of the lemma:** The lemma is proved by induction on $k$. The case $k = 1$ is trivial as $(1 - x)^1 = 1 - 1 \cdot x$. Now assume that the inequality $(1 - x)^k \geq 1 - kx$ holds for some $k \in \mathbb{N}$ and all $x \in (0, 1)$. Then $(1 - x)^{k+1} = (1 - x)^k(1 - x) \geq (1 - kx)(1 - x) = 1 - kx - x + kx^2 > 1 - (k + 1)x. \quad \blacksquare$

**Remark.** According to the Binomial Formula,

$$(1 - x)^k = 1 - kx + \frac{k(k-1)}{2}x^2 - \ldots$$
Now let us show that the sequence \( \{x_n\} \) is bounded. Since \( \{x_n\} \) is increasing, it is enough to show that it is bounded above. By the Binomial Formula,

\[
x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k .
\]

Observe that \( \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k \leq 1 \) for all \( k, \ 0 \leq k \leq n \).

It follows that \( x_n \leq \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \).

Further observe that \( k! \geq 2^{k-1} \) for all \( k \geq 0 \). Therefore we obtain

\[
x_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3.
\]
**Cauchy sequences**

*Definition.* A sequence \( \{x_n\} \) of real numbers is called a **Cauchy sequence** if for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_n - x_m| < \varepsilon \) whenever \( n, m \geq N \).

**Theorem**  Any convergent sequence is Cauchy.

*Proof:* Let \( \{x_n\} \) be a convergent sequence and \( a \) be its limit. Then for any \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_n - a| < \varepsilon/2 \) whenever \( n \geq N \). Now for any natural numbers \( n, m \geq N \) we have

\[
|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Thus \( \{x_n\} \) is a Cauchy sequence.
Theorem  Any Cauchy sequence is convergent.

Proof: Suppose \( \{x_n\} \) is a Cauchy sequence. First let us show that this sequence is bounded. Since \( \{x_n\} \) is Cauchy, there exists \( N \in \mathbb{N} \) such that \( |x_n - x_m| < 1 \) whenever \( n, m \geq N \). In particular, \( |x_n - x_N| < 1 \) for all \( n \geq N \). Then
\[
|x_n| = |(x_n - x_N) + x_N| \leq |x_n - x_N| + |x_N| < |x_N| + 1.
\]
It follows that for any \( n \in \mathbb{N} \) we have \( |x_n| \leq M \), where \( M = \max(|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1) \).

Now the Bolzano-Weierstrass theorem implies that \( \{x_n\} \) has a subsequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) converging to some \( a \in \mathbb{R} \). Given \( \varepsilon > 0 \), there exists \( K_\varepsilon \in \mathbb{N} \) such that \( |x_{n_k} - a| < \varepsilon/2 \) for all \( k \geq K_\varepsilon \). Also, there exists \( N_\varepsilon \in \mathbb{N} \) such that \( |x_n - x_m| < \varepsilon/2 \) whenever \( n, m \geq N_\varepsilon \). Let \( k = \max(K_\varepsilon, N_\varepsilon) \). Then \( k \geq K_\varepsilon \) and \( n_k \geq k \geq N_\varepsilon \). Therefore for any \( n \geq N_\varepsilon \) we obtain
\[
|x_n - a| = |(x_n - x_{n_k}) + (x_{n_k} - a)| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Thus the entire sequence \( \{x_n\} \) converges to \( a \).
Limit points

Definition. A limit point of a sequence \( \{x_n\} \) is the limit of any convergent subsequence of \( \{x_n\} \).

Examples and properties.

- A convergent sequence has only one limit point, its limit.
- Any bounded sequence has at least one limit point.
- If a bounded sequence is not convergent, then it has at least two limit points.
- The sequence \( \{(-1)^n\} \) has two limit points, 1 and \(-1\).
- If all elements of a sequence belong to a closed interval \([a, b]\), then all its limit points belong to \([a, b]\) as well.
- The set of limit points of the sequence \( \{\sin n\} \) is the entire interval \([-1, 1]\).
- If a sequence diverges to infinity, then it has no limit points.
- If a sequence does not diverge to infinity, then it has a bounded subsequence and hence it has a limit point.