MATH 409
Advanced Calculus I

Lecture 25:
Review for the final exam.
Topics for the final

Part I: Axiomatic model of the real numbers

- Axioms of an ordered field
- Completeness axiom
- Archimedean principle
- Principle of mathematical induction
- Binomial formula
- Countable and uncountable sets

Wade’s book: 1.1–1.6, Appendix A
Topics for the final

Part II: Limits and continuity

• Limits of sequences
• Limit theorems for sequences
• Monotone sequences
• Bolzano-Weierstrass theorem
• Cauchy sequences
• Limits of functions
• Limit theorems for functions
• Continuity of functions
• Extreme value and intermediate value theorems
• Uniform continuity

Wade’s book: 2.1–2.5, 3.1–3.4
Topics for the final

Part III-a: Differential calculus

- Derivative of a function
- Differentiability theorems
- Derivative of the inverse function
- The mean value theorem
- Taylor’s formula
- l’Hôpital’s rule

Wade’s book: 4.1–4.5
Topics for the final

*Part III-b: Integral calculus*

- Darboux sums, Riemann sums, the Riemann integral
- Properties of integrals
- The fundamental theorem of calculus
- Integration by parts
- Change of the variable in an integral
- Improper integrals, absolute integrability

*Wade’s book: 5.1–5.4*
Topics for the final

Part IV: Infinite series

- Convergence of series
- Comparison test and integral test
- Alternating series test
- Absolute convergence
- Ratio test and root test

Wade’s book: 6.1–6.4
Archimedean Principle  For any real number $\varepsilon > 0$ there exists a natural number $n$ such that $n\varepsilon > 1$.

Theorem  The sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N} \times \mathbb{N}$ are countable.

Theorem  The set $\mathbb{R}$ is uncountable.
Theorems on limits

**Squeeze Theorem**  If \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a \) and \( x_n \leq w_n \leq y_n \) for all sufficiently large \( n \), then \( \lim_{n \to \infty} w_n = a \).

**Theorem**  Any monotone sequence converges to a limit if bounded, and diverges to infinity otherwise.

**Theorem**  Any Cauchy sequence is convergent.
Theorems on derivatives

**Theorem**  If functions $f$ and $g$ are differentiable at a point $a \in \mathbb{R}$, then their sum $f + g$, difference $f - g$, and product $f \cdot g$ are also differentiable at $a$. Moreover,

$$
(f + g)'(a) = f'(a) + g'(a),
$$

$$
(f - g)'(a) = f'(a) - g'(a),
$$

$$
(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a).
$$

If, additionally, $g(a) \neq 0$ then the quotient $f/g$ is also differentiable at $a$ and

$$
\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}.
$$

**Mean Value Theorem**  If a function $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. 


Theorems on integrals

**Theorem**  If functions $f, g$ are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and
\[
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

**Theorem**  If a function $f$ is integrable on $[a, b]$ then for any $c \in (a, b)$,
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]
**Theorems on series**

**Integral Test** Suppose that $f : [1, \infty) \to \mathbb{R}$ is positive and decreasing on $[1, \infty)$. Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if the function $f$ is improperly integrable on $[1, \infty)$.

**Ratio Test** Let $\{a_n\}$ be a sequence of reals with $a_n \neq 0$ for large $n$. Suppose that a limit

$$r = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

exists (finite or infinite).

**(i)** If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

**(ii)** If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
Sample problems for the final exam

Problem 1 (20 pts.) Suppose $E_1, E_2, E_3, \ldots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup E_3 \cup \ldots$ is also a countable set.

Problem 2 (20 pts.) Find the following limits:

(i) $\lim_{x \to 0} \log \frac{1}{1 + \cot(x^2)}$,   
(ii) $\lim_{x \to 64} \frac{\sqrt{x - 8}}{\sqrt[3]{x - 4}}$,   
(iii) $\lim_{n \to \infty} \left(1 + \frac{c}{n}\right)^n$, where $c \in \mathbb{R}$. 
Sample problems for the final exam

Problem 3 (20 pts.) Prove that the series
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]
converges to \( \sin x \) for any \( x \in \mathbb{R} \).

Problem 4 (20 pts.) Find an indefinite integral and evaluate definite integrals:

(i) \( \int \frac{\sqrt{1 + \sqrt{4x}}}{2\sqrt{x}} \, dx \),

(ii) \( \int_{0}^{\sqrt{3}} \frac{x^2 + 6}{x^2 + 9} \, dx \),

(iii) \( \int_{0}^{\infty} x^2 e^{-x} \, dx \).
Problem 5 (20 pts.) For each of the following series, determine whether the series converges and whether it converges absolutely:

(i) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} , \]

(ii) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n} + 2^n \cos n}{n!} , \]

(iii) \[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n} . \]
Sample problems for the final exam

Bonus Problem 6 (15 pts.) Prove that an infinite product

$$\prod_{n=1}^{\infty} \frac{n^2 + 1}{n^2} = \frac{2}{1} \cdot \frac{5}{4} \cdot \frac{10}{9} \cdot \frac{17}{16} \cdot \ldots$$

converges, that is, partial products \( \prod_{k=1}^{n} \frac{k^2+1}{k^2} \) converge to a finite limit as \( n \to \infty \).
Problem 1. Suppose $E_1, E_2, E_3, \ldots$ are countable sets. Prove that their union $E_1 \cup E_2 \cup \ldots$ is also a countable set.

First we are going to show that the set $\mathbb{N} \times \mathbb{N}$ is countable. Consider a relation $\preceq$ on the set $\mathbb{N} \times \mathbb{N}$ such that $(n_1, n_2) \preceq (m_1, m_2)$ if and only if either $n_1 + n_2 < m_1 + m_2$ or else $n_1 + n_2 = m_1 + m_2$ and $n_1 < m_1$. It is easy to see that $\preceq$ is a strict linear order. Moreover, for any pair $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ there are only finitely many pairs $(n_1, n_2)$ such that $(n_1, n_2) \preceq (m_1, m_2)$. It follows that $\preceq$ is a well-ordering. Now we define inductively a mapping $F : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ such that for any $n \in \mathbb{N}$ the pair $F(n)$ is the least (relative to $\preceq$) pair different from $F(k)$ for all natural numbers $k < n$. It follows from the construction that $F$ is bijective. The inverse mapping $F^{-1}$ can be given explicitly by

$$F^{-1}(n_1, n_2) = \frac{(n_1 + n_2 - 2)(n_1 + n_2 - 1)}{2} + n_1, \quad n_1, n_2 \in \mathbb{N}.$$ 

Thus $\mathbb{N} \times \mathbb{N}$ is a countable set.
Now suppose that $E_1, E_2, \ldots$ are countable sets. Then for any $n \in \mathbb{N}$ there exists a bijective mapping $f_n : \mathbb{N} \to E_n$. Let us define a map $g : \mathbb{N} \times \mathbb{N} \to E_1 \cup E_2 \cup \ldots$ by $g(n_1, n_2) = f_{n_1}(n_2)$. Obviously, $g$ is onto.

Since the set $\mathbb{N} \times \mathbb{N}$ is countable, there exists a sequence $p_1, p_2, p_3, \ldots$ that forms a complete list of its elements. Then the sequence $g(p_1), g(p_2), g(p_3), \ldots$ contains all elements of the union $E_1 \cup E_2 \cup E_3 \cup \ldots$. Although the latter sequence may include repetitions, we can choose a subsequence \{g(p_{n_k})\} in which every element of the union appears exactly once. Note that the subsequence is infinite since each of the sets $E_1, E_2, \ldots$ is infinite.

Now the map $h : \mathbb{N} \to E_1 \cup E_2 \cup E_3 \cup \ldots$ defined by $h(k) = g(p_{n_k}), \ k = 1, 2, \ldots$, is a bijection.
Problem 2. Find the following limits:

(i) \( \lim_{x \to 0} \log \frac{1}{1 + \cot(x^2)} \).

The function \( f(x) = \log \frac{1}{1 + \cot(x^2)} \) can be represented as the composition of 4 functions: \( f_1(x) = x^2 \), \( f_2(y) = \cot y \), \( f_3(z) = (1 + z)^{-1} \), and \( f_4(u) = \log u \).

Since the function \( f_1 \) is continuous, we have \( \lim_{x \to 0} f_1(x) = f_1(0) = 0 \). Moreover, \( f_1(x) > 0 \) for \( x \neq 0 \).

Since \( \lim_{y \to 0^+} \cot y = +\infty \), it follows that \( f_2(f_1(x)) \to +\infty \) as \( x \to 0 \).

Further, \( f_3(z) \to 0^+ \) as \( z \to +\infty \) and \( f_4(u) \to -\infty \) as \( u \to 0^+ \). Finally, \( f(x) = f_4(f_3(f_2(f_1(x)))) \to -\infty \) as \( x \to 0 \).
Problem 2. Find the following limits:

(ii) \( \lim_{{x \to 64}} \frac{\sqrt[3]{x} - 8}{\sqrt[3]{x} - 4} \).

Consider a function \( u(x) = x^{1/6} \) defined on \((0, \infty)\). Since this function is continuous at 64 and \( u(64) = 2 \), we obtain

\[
\lim_{{x \to 64}} \frac{\sqrt[3]{x} - 8}{\sqrt[3]{x} - 4} = \lim_{{x \to 64}} \frac{(u(x))^3 - 8}{(u(x))^2 - 4}
\]

\[
= \lim_{{y \to 2}} \frac{y^3 - 8}{y^2 - 4} = \lim_{{y \to 2}} \frac{(y - 2)(y^2 + 2y + 4)}{(y - 2)(y + 2)}
\]

\[
= \lim_{{y \to 2}} \frac{y^2 + 2y + 4}{y + 2} = \frac{y^2 + 2y + 4}{y + 2} \bigg|_{{y=2}} = 3.
\]
Problem 2. Find the following limits:

(iii) \( \lim_{n \to \infty} \left(1 + \frac{c}{n}\right)^n \), where \( c \in \mathbb{R} \).

Let \( a_n = (1 + c/n)^n \), \( n = 1, 2, \ldots \). For \( n \) large enough, we have \( 1 + c/n > 0 \) so that \( a_n > 0 \). Then

\[
\log a_n = \log \left(1 + \frac{c}{n}\right)^n = n \log \left(1 + \frac{c}{n}\right) = \frac{\log(1 + cx)}{x} \bigg|_{x=1/n}.
\]

Since \( 1/n \to 0 \) as \( n \to \infty \) and

\[
\lim_{x \to 0} \frac{\log(1 + cx)}{x} = \left(\log(1 + cx)\right)' \bigg|_{x=0} = \frac{c}{1 + cx} \bigg|_{x=0} = c,
\]

we obtain that \( \log a_n \to c \) as \( n \to \infty \). Therefore \( a_n = e^{\log a_n} \to e^c \) as \( n \to \infty \).
Problem 3. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

converges to \( \sin x \) for any \( x \in \mathbb{R} \).

The function \( f(x) = \sin x \) is infinitely differentiable on \( \mathbb{R} \). According to Taylor’s formula, for any \( x, x_0 \in \mathbb{R} \) and \( n \in \mathbb{N} \),

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x, x_0),$$

where \( R_n(x, x_0) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1} \) for some \( \theta = \theta(x, x_0) \) between \( x \) and \( x_0 \). Since \( f'(x) = \cos x \) and \( f''(x) = -\sin x = -f(x) \) for all \( x \in \mathbb{R} \), it follows that \( |f^{(n+1)}(\theta)| \leq 1 \) for all \( n \in \mathbb{N} \) and \( \theta \in \mathbb{R} \). Further, one derives that \( R_n(x, x_0) \to 0 \) as \( n \to \infty \). Thus we obtain an expansion of \( \sin x \) into a series. In the case \( x_0 = 0 \), this is the required series (up to zero terms).
Problem 4. Find an indefinite integral and evaluate definite integrals:

(i) \( \int \frac{\sqrt{1 + \frac{4}{x}}}{2\sqrt{x}} \, dx \).

To find this integral, we change the variable twice. First

\[
\int \frac{\sqrt{1 + \frac{4}{x}}}{2\sqrt{x}} \, dx = \int \sqrt{1 + \frac{4}{x}} \, (\sqrt{x})' \, dx = \int \sqrt{1 + \sqrt{u}} \, du,
\]

where \( u = \sqrt{x} \). Secondly, we introduce a variable \( w = \sqrt{1 + \sqrt{u}} \). Then \( u = (w^2 - 1)^2 \) so that

\[
du = \left( (w^2 - 1)^2 \right)' \, dw = 2(w^2 - 1) \cdot 2w \, dw = (4w^3 - 4w) \, dw.
\]

Consequently,

\[
\int \sqrt{1 + \sqrt{u}} \, du = \int w \, du = \int (4w^4 - 4w^2) \, dw
\]

\[
= \frac{4}{5} w^5 - \frac{4}{3} w^3 + C = \frac{4}{5} (1 + x^{1/4})^{5/2} - \frac{4}{3} (1 + x^{1/4})^{3/2} + C.
\]
Problem 4. Find an indefinite integral and evaluate definite integrals:

\[ (ii) \int_{0}^{\sqrt{3}} \frac{x^2 + 6}{x^2 + 9} \, dx. \]

To evaluate this definite integral, we use linearity of the integral and a substitution \( x = 3u \):

\[
\int_{0}^{\sqrt{3}} \frac{x^2 + 6}{x^2 + 9} \, dx = \int_{0}^{\sqrt{3}} \left( 1 - \frac{3}{x^2 + 9} \right) \, dx = \int_{0}^{\sqrt{3}} 1 \, dx
\]

\[
- \int_{0}^{\sqrt{3}} \frac{3}{x^2 + 9} \, dx = \sqrt{3} - \int_{0}^{\sqrt{3}/3} \frac{3}{(3u)^2 + 9} \, d(3u)
\]

\[
= \sqrt{3} - \int_{0}^{1/\sqrt{3}} \frac{1}{u^2 + 1} \, du = \sqrt{3} - \arctan u \bigg|_{u=0}^{1/\sqrt{3}} = \sqrt{3} - \frac{\pi}{6}.
\]
Problem 4. Find an indefinite integral and evaluate definite integrals:

(iii) \( \int_{0}^{\infty} x^2 e^{-x} \, dx \).

To evaluate the improper integral, we integrate by parts twice:

\[
\begin{align*}
\int_{0}^{\infty} x^2 e^{-x} \, dx &= - \int_{0}^{\infty} x^2 (e^{-x})' \, dx = - \int_{0}^{\infty} x^2 \, d(e^{-x}) \\
&= -x^2 e^{-x}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-x} \, d(x^2) = \int_{0}^{\infty} e^{-x} (x^2)' \, dx \\
&= \int_{0}^{\infty} 2xe^{-x} \, dx = - \int_{0}^{\infty} 2x(e^{-x})' \, dx = - \int_{0}^{\infty} 2x \, d(e^{-x}) \\
&= -2xe^{-x}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-x} \, d(2x) = \int_{0}^{\infty} 2e^{-x} \, dx \\
&= -2e^{-x}\bigg|_{0}^{\infty} = 2.
\end{align*}
\]
Problem 5. For each of the following series, determine if the series converges and if it converges absolutely:

(i) \( \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \), (ii) \( \sum_{n=1}^{\infty} \frac{\sqrt{n} + 2^n \cos n}{n!} \), (iii) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n} \).

The first series diverges since

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{n+1} + \sqrt{n})^2} > \sum_{n=1}^{\infty} \frac{1}{4(n+1)} = +\infty.
\]

The second series can be represented as \( \sum_{n=1}^{\infty} (b_n + c_n \cos n) \), where \( b_n = \sqrt{n}/n! \) and \( c_n = 2^n/n! \) for all \( n \in \mathbb{N} \). The series \( \sum_{n=1}^{\infty} b_n \) and \( \sum_{n=1}^{\infty} c_n \) both converge (due to the Ratio Test), and so does \( \sum_{n=1}^{\infty} (b_n + c_n) \). Since \( |b_n + c_n \cos n| \leq b_n + c_n \) for all \( n \in \mathbb{N} \), the series \( \sum_{n=1}^{\infty} (b_n + c_n \cos n) \) converges absolutely due to the Comparison Test.

Finally, the third series converges (due to the Alternating Series Test), but not absolutely (due to the Integral Test).