Test 1: Solutions

Problem 1 (20 pts.) Prove that for any \( x \in (0, 1) \) and any natural number \( n \),
\[
(1 - x)^n \leq 1 - nx + \frac{n(n - 1)}{2} x^2.
\]
The proof is by induction on \( n \). First we consider the case \( n = 1 \). In this case the inequality reduces to \((1 - x)^1 \leq 1 - x\), which is true for any \( x \in \mathbb{R} \). Now assume that the inequality holds for \( n = k \), that is,
\[
(1 - x)^k \leq 1 - kx + \frac{k(k - 1)}{2} x^2
\]
for all \( x \in (0, 1) \). Multiplying both sides of this inequality by the positive number \( 1 - x \), we get
\[
(1 - x)^{k+1} \leq 1 - (k+1)x + \frac{(k+1)k}{2} x^2 - \frac{k(k-1)}{2} x^3.
\]
Since \( x > 0 \), it follows that
\[
(1 - x)^{k+1} \leq 1 - (k+1)x + \frac{(k+1)k}{2} x^2,
\]
which means that the inequality holds for \( n = k + 1 \) as well. By induction, the inequality holds for every natural number \( n \) and every \( x \in (0,1) \).

Problem 2 (25 pts.) For each of the following sequences, find the limit or show that the sequence is not convergent.

(i) \( x_n = \sqrt{n+3} - \sqrt{n-1}, \ n \in \mathbb{N} \).

(ii) \( y_1 = 17, \ y_n = 1 + \sqrt{y_{n-1} - 1} \) for \( n \geq 2 \).

(ii) \( z_n = n^3 \sin \frac{1}{n^2}, \ n \in \mathbb{N} \).

Solution: \( \lim_{n \to \infty} x_n = 0, \lim_{n \to \infty} y_n = 2 \). The sequence \( \{z_n\} \) diverges to \( +\infty \).

For any \( n \in \mathbb{N} \) we have
\[
0 < x_n = \sqrt{n+3} - \sqrt{n-1} = \frac{(\sqrt{n+3} - \sqrt{n-1})(\sqrt{n+3} + \sqrt{n-1})}{\sqrt{n+3} + \sqrt{n-1}} = \frac{4}{\sqrt{n+3} + \sqrt{n-1}} < \frac{4}{\sqrt{n+3}},
\]
which implies that \( x_n \to 0 \) as \( n \to \infty \). Further, \( y_n - 1 = \sqrt{y_{n-1} - 1} \) for all \( n \geq 2 \). It follows by induction that \( y_n - 1 = (y_1 - 1)^{(1/2)^{n-1}} = 16^{(1/2)^{n-1}} = 16^{2^{-n}} \) for all \( n \in \mathbb{N} \). Since \( 16^{1/m} \to 1 \) as \( m \to \infty \), we obtain that \( y_n - 1 \to 1 \) as \( n \to \infty \). Then \( y_n \to 2 \) as \( n \to \infty \).
Finally, we know that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). As a consequence, \( n^2 \sin(1/n^2) = \sin(1/n^2)/(1/n^2) \to 1 \) as \( n \to \infty \). In particular, \( n^2 \sin(1/n^2) > 1/2 \) for all sufficiently large \( n \). Then \( z_n = n^3 \sin(1/n^2) > n/2 \) for all sufficiently large \( n \), which implies that \( z_n \to +\infty \) as \( n \to \infty \).

**Problem 3 (25 pts.)** Prove the Bolzano-Weierstrass Theorem: any bounded sequence of real numbers has a convergent subsequence.

Suppose \( \{x_n\} \) is a bounded sequence of real numbers. We are going to build a nested sequence of intervals \( I_n = [a_n, b_n] \), \( n = 1, 2, \ldots \), such that each \( I_n \) contains infinitely many elements of \( \{x_n\} \) and \( |I_{n+1}| = |I_n|/2 \) for all \( n \in \mathbb{N} \). The sequence is built inductively. First we set \( I_1 \) to be any closed bounded interval that contains all elements of \( \{x_n\} \) (such an interval exists since the sequence \( \{x_n\} \) is bounded). Now assume that for some \( n \in \mathbb{N} \) the interval \( I_n \) is already chosen and it contains infinitely many elements of the sequence \( \{x_n\} \). Then at least one of the subintervals \( I' = [a_n, (a_n + b_n)/2] \) and \( I'' = [(a_n + b_n)/2, b_n] \) also contains infinitely many elements of \( \{x_n\} \). We set \( I_{n+1} \) to be such a subinterval. By construction, \( I_{n+1} \subset I_n \) and \( |I_{n+1}| = |I_n|/2 \). Since \( |I_{n+1}| = |I_n|/2 \) for all \( n \in \mathbb{N} \), it follows by induction that \( |I_n| = |I_1|/2^n \) for all \( n \in \mathbb{N} \). As a consequence, \( |I_n| \to 0 \) as \( n \to \infty \). By the nested intervals property, the intersection \( I_1 \cap I_2 \cap I_3 \cap \ldots \) consists of a single number \( c \).

Next we are going to build a strictly increasing sequence of natural numbers \( n_1, n_2, \ldots \) such that \( x_{n_k} \in I_k \) for all \( k \in \mathbb{N} \). The sequence is built inductively. First we choose \( n_1 \) so that \( x_{n_1} \in I_1 \). Now assume that for some \( k \in \mathbb{N} \) the number \( n_k \) is already chosen. Since the interval \( I_{k+1} \) contains infinitely many elements of the sequence \( \{x_n\} \), there exists \( m > n_k \) such that \( x_m \in I_{k+1} \). We set \( n_{k+1} = m \).

Now we claim that the subsequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) of the sequence \( \{x_n\} \) converges to \( c \). Indeed, for any \( k \in \mathbb{N} \) the points \( x_{n_k} \) and \( c \) both belong to the interval \( I_k \). Hence \( |x_{n_k} - c| \leq |I_k| \). Since \( |I_k| \to 0 \) as \( k \to \infty \), it follows that \( x_{n_k} \to c \) as \( k \to \infty \).

**Problem 4 (20 pts.)** Consider a function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
 f(x) = \begin{cases} 
 \sin \frac{2\pi}{x} & \text{if } -1 < x < 0, \\
 |x + 1| - |x - 1| & \text{if } 0 \leq x < 1, \\
 e^{-x} \sin(\pi x) & \text{if } |x| \geq 1. 
\end{cases}
\]

(i) Determine all points at which the function \( f \) is continuous.
(ii) Determine if the limits \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \) exist (limits may be finite or infinite).

**Solution:** The function \( f \) is continuous on \( \mathbb{R} \setminus \{0, 1\} \) and discontinuous at 0 and 1. \( \lim_{x \to +\infty} f(x) = 0 \). There is no limit (finite or infinite) of \( f(x) \) as \( x \to -\infty \).

The function \( g_1(x) = 2\pi/x \) is continuous on \( \mathbb{R} \setminus \{0\} \) while the function \( g_2(x) = \sin x \) is continuous on \( \mathbb{R} \). It follows that the composition \( g(x) = g_2(g_1(x)) = \sin(2\pi/x) \) is continuous on \( \mathbb{R} \setminus \{0\} \). Note that the left-hand limit of \( g \) at 0 does not exists. Indeed, let \( x_n = -4/(4n + 1) \) and \( y_n = -4/(4n - 1) \) for all \( n \in \mathbb{N} \). Then \( \{x_n\} \) and \( \{y_n\} \) are two sequences of negative numbers converging to 0. We have
\( g(x_n) = \sin(2\pi/x_n) = \sin(-\pi/2 - 2\pi n) = -1 \) and \( g(y_n) = \sin(2\pi/y_n) = \sin(\pi/2 - 2\pi n) = 1 \) for all \( n \in \mathbb{N} \). It follows that there is no limit of \( g(x) \) as \( x \to 0^- \).

Further, the functions \( h_1(x) = x + 1, h_2(x) = x - 1, \) and \( h_3(x) = |x| \) are continuous on \( \mathbb{R} \). Hence the composition functions \( h_4(x) = h_3(h_1(x)) = |x + 1| \) and \( h_5(x) = h_3(h_2(x)) = |x - 1| \) are also continuous on \( \mathbb{R} \) as well as the sum \( h(x) = h_4(x) + h_5(x) = |x + 1| + |x - 1| \).

Finally, the functions \( j_1(x) = -x, j_2(x) = e^x, j_3(x) = \pi x, \) and \( j_4(x) = \sin x \) are continuous on \( \mathbb{R} \). Therefore the compositions \( j_5(x) = j_2(j_1(x)) = e^{-x} \) and \( j_6(x) = j_4(j_3(x)) = \sin(\pi x) \) are continuous on \( \mathbb{R} \) as well as the product \( j(x) = j_5(x)j_6(x) = e^{-x} \sin(\pi x) \).

By definition, the function \( f \) coincides with \( g \) on \((-1, 0)\), with \( h \) on \([0, 1)\), and with \( j \) on the union \((-\infty, 1] \cup [1, \infty)\). Hence it follows from the above that \( f \) is continuous on \( \mathbb{R} \setminus \{-1, 0, 1\} \). Moreover, \( f \) has no left-hand limit at 0 since \( g \) does not have such a limit. Therefore \( f \) is discontinuous at 0. It remains to determine whether \( f \) is continuous at 1 and \(-1\). We have

\[
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} j(x) = j(1) = 0, \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} h(x) = h(1) = 2, \quad f(1) = 0,
\]

\[
\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} g(x) = g(-1) = 0, \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} j(x) = j(-1) = 0, \quad f(-1) = 0.
\]

Thus \( f \) is continuous at \(-1\) and has a jump discontinuity at 1.

Since \( f(x) = j(x) \) for \(|x| \geq 1\), the behaviour of the function \( f \) at infinity is the same as that of \( j(x) = e^{-x} \sin(\pi x) \). We know that \(|\sin(\pi x)| \leq 1 \) and \( e^{-x} \to 0 \) as \( x \to +\infty \). It follows that \( j(x) \to 0 \) as \( x \to +\infty \). To show that \( \lim_{x \to -\infty} j(x) \) does not exist, consider two sequences \( X_n = -n \) and \( Y_n = 1/2 - 2n, n \in \mathbb{N} \). Both sequences diverge to \(-\infty\). However \( j(X_n) = 0 \) for all \( n \) while \( j(Y_n) = e^{-Y_n} \sin(\pi Y_n) = e^{1/2+2n} \to +\infty \) as \( n \to \infty \).

**Bonus Problem 5 (15 pts.)** Suppose \( E \) is an uncountable subset of \( \mathbb{R} \). Prove that for any \( M > 0 \) one can find distinct elements \( x_1, x_2, \ldots, x_n \in E \) such that \(|x_1 + x_2 + \cdots + x_n| > M\).

For any \( \varepsilon > 0 \) let \( A_\varepsilon = E \cap (\varepsilon, +\infty) \) and \( B_\varepsilon = E \cap (-\infty, -\varepsilon) \). If \( x_1, x_2, \ldots, x_n \) are distinct elements of \( A_\varepsilon \), then \( x_1 + x_2 + \cdots + x_n > n\varepsilon \). In the case \( A_\varepsilon \) is an infinite set, we can find arbitrarily many such elements so that \( n\varepsilon \) can be made larger than any prescribed \( M > 0 \). Similarly, if the set \( B_\varepsilon \) is infinite, we can find distinct elements \( x_1, x_2, \ldots, x_n \in B_\varepsilon \) such that \(|x_1 + x_2 + \cdots + x_n| > n\varepsilon \).

It remains to show that for some \( \varepsilon > 0 \) at least one of the sets \( A_\varepsilon \) and \( B_\varepsilon \) is infinite. Indeed, the set \( E \) can be represented as the union of countably many sets:

\[
E = (E \cap \{0\}) \cup A_1 \cup B_1 \cup A_{1/2} \cup B_{1/2} \cup \cdots \cup A_{1/n} \cup B_{1/n} \cup \ldots
\]

Since the countable union of finite sets is at most countable, it follows that for some \( n \in \mathbb{N} \) one of the sets \( A_{1/n} \) and \( B_{1/n} \) is infinite.