Test 2: Solutions

Problem 1 (20 pts.) Find \( \min_{x>0} x^{3x} \).

Solution: \( \min_{x>0} x^{3x} = e^{-3/e} \).

The function \( f(x) = x^{3x} \) is well defined and positive on \((0, \infty)\). Hence \( f(x) = e^{\log f(x)} = e^{\log x^{3x}} = e^{3x \log x} \) for all \( x > 0 \). Now it follows from the Chain Rule and the Product Rule that \( f \) is differentiable on \((0, \infty)\) and

\[
f'(x) = e^{3x \log x} (3x \log x)' = x^{3x} \left( (3x)' \log x + 3x (\log x)' \right) = x^{3x} (3 \log x + 3) = 3x^{3x} (\log x + 1)
\]

for all \( x > 0 \). Notice that a function \( g(x) = \log x + 1 \) is strictly increasing on \((0, \infty)\). Since \( g(1/e) = 0 \), this implies that \( g(x) > 0 \) for \( x > 1/e \) and \( g(x) < 0 \) for \( 0 < x < 1/e \). Consequently, \( f'(x) > 0 \) for \( x > 1/e \) and \( f'(x) < 0 \) for \( 0 < x < 1/e \). It follows that the function \( f \) is strictly decreasing on the interval \((0, 1/e)\) and strictly increasing on the interval \([1/e, \infty)\). In particular, \( f(x) > f(1/e) \) for each \( x > 0, x \neq 1/e \). Thus \( \min_{x>0} f(x) = f(1/e) = (1/e)^{3/e} = e^{-3/e} \).

Problem 2 (25 pts.) Consider a function \( f : \mathbb{R} \to \mathbb{R} \) given by

\[
f(x) = \begin{cases} 
\frac{\sin x}{x} & \text{if } x \neq 0, \\
1 & \text{if } x = 0.
\end{cases}
\]

(i) Prove that \( f \) is differentiable at 0 and find \( f'(0) \).
(ii) Prove that \( f \) is twice differentiable at 0 and find \( f''(0) \).

Solution: \( f'(0) = 0, f''(0) = -1/3 \).

For any \( x \neq 0 \),

\[
\frac{f(x) - f(0)}{x - 0} = \frac{1}{x} \left( \frac{\sin x}{x} - 1 \right) = \frac{\sin x - x}{x^2}.
\]

Functions \( h_1(x) = \sin x - x \) and \( h_2(x) = x^2 \) are infinitely differentiable on \( \mathbb{R} \). Since \( h_1(0) = h_2(0) = 0 \), we have \( \lim_{x \to 0} h_1(x) = \lim_{x \to 0} h_2(x) = 0 \). Further, \( h_1'(x) = \cos x - 1 \) and \( h_2'(x) = 2x \) for all \( x \in \mathbb{R} \). In particular, \( h_1'(0) = h_2'(0) = 0 \) so that \( \lim_{x \to 0} h_1'(x) = \lim_{x \to 0} h_2'(x) = 0 \). Further, \( h_1''(x) = -\sin x \) and \( h_2''(x) = 2 \) for all \( x \in \mathbb{R} \). In particular, \( h_1''(0) = 0 \) so that \( \lim_{x \to 0} h_1''(x) = 0 \). Finally, using l'Hôpital’s Rule (twice) and a limit theorem, we obtain

\[
f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x \to 0} \frac{h_1'(x)}{h_2'(x)} = \lim_{x \to 0} \frac{h_1''(x)}{h_2''(x)} = \lim_{x \to 0} \frac{h_1''(x)}{2} = 0 = 0.
\]
Thus $f$ is differentiable at 0. As a consequence, $f$ is continuous at 0, that is,
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
By the Quotient Rule, the function $f$ is also differentiable at any $x \neq 0$ and
\[
f'(x) = \left(\frac{\sin x}{x}\right)' = \frac{(\sin x)' \cdot x - \sin x \cdot x'}{x^2} = \frac{x \cos x - \sin x}{x^2}.
\]
Notice that $\lim_{x \to 0} (x \cos x - \sin x) = 0$. Using l'Hôpital’s Rule, we obtain
\[
f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2} = \lim_{x \to 0} \frac{(x \cos x - \sin x)'}{(x^3)'}
\]
\[
= \lim_{x \to 0} \frac{-x \sin x}{3x^2} = -\frac{1}{3} \lim_{x \to 0} \frac{\sin x}{x} = -\frac{1}{3}.
\]

**Problem 3 (20 pts.)** Prove the following version of the Fundamental Theorem of Calculus: if a function $F$ is differentiable on $[a, b]$ and the derivative $F'$ is integrable on $[a, b]$, then
\[
\int_a^b F'(t) \, dt = F(b) - F(a).
\]
Consider an arbitrary partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Let us choose samples $t_j \in [x_{j-1}, x_j]$ for the Riemann sum $S(F', P, t_j)$ so that $F(x_j) - F(x_{j-1}) = F'(t_j) (x_j - x_{j-1})$, $j = 1, 2, \ldots, n$ (this is possible due to the Mean Value Theorem). Then
\[
S(F', P, t_j) = \sum_{j=1}^n F'(t_j) (x_j - x_{j-1}) = \sum_{j=1}^n (F(x_j) - F(x_{j-1})) = F(x_n) - F(x_0) = F(b) - F(a).
\]
Since the sums $S(F', P, t_j)$ should converge to the integral $\int_a^b F'(t) \, dt$ as $\|P\| \to 0$, the theorem follows.

**Problem 4 (25 pts.)** Find an indefinite integral and evaluate definite integrals:
\[
(i) \int x^2 \sin x \, dx, \quad (ii) \int_0^{1/2} \frac{2}{1 - x^2} \, dx, \quad (iii) \int_0^{1/2} \sqrt{1 - x^2} \, dx.
\]
**Solution:**
\[
\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C, \quad \int_0^{1/2} \frac{2}{1 - x^2} \, dx = \log 3,
\]
\[
\int_0^{1/2} \sqrt{1 - x^2} \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.
\]
To find the indefinite integral of the function $y(x) = x^2 \sin x$, we integrate by parts twice:

$$
\int x^2 \sin x \, dx = - \int x^2 (\cos x)' \, dx = - \int x^2 d(\cos x) = -x^2 \cos x + \int \cos x \, dx
$$

$$
= -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2 \int x \, d(\sin x)
$$

$$
= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.
$$

To integrate a rational function $y(x) = \frac{2}{1 - x^2}$, we expand it into a sum of simple fractions:

$$
\frac{2}{1 - x^2} = \frac{2}{(1 - x)(1 + x)} = \frac{1}{1 + x} + \frac{1}{1 - x}.
$$

Then

$$
\int_{0}^{1/2} \frac{2}{1 - x^2} \, dx = \int_{0}^{1/2} \frac{1}{1 + x} \, dx + \int_{0}^{1/2} \frac{1}{1 - x} \, dx = \log(1 + x)|_{x=0}^{1/2} - \log(1 - x)|_{x=0}^{1/2}
$$

$$
= \left( \log \frac{3}{2} - \log 1 \right) - \left( \log \frac{1}{2} - \log 1 \right) = \log 3.
$$

To integrate the function $y(x) = \sqrt{1 - x^2}$, we use a substitution $x = \sin t$ (observe that $x$ changes from 0 to 1/2 when $t$ changes from 0 to $\pi/6$):

$$
\int_{0}^{1/2} \sqrt{1 - x^2} \, dx = \int_{0}^{\pi/6} \sqrt{1 - \sin^2 t} \, d(\sin t) = \int_{0}^{\pi/6} \sqrt{1 - \sin^2 t} \,(\sin t)' \, dt
$$

$$
= \int_{0}^{\pi/6} \cos^2 t \cdot \cos t \, dt = \int_{0}^{\pi/6} \cos^2 t \, dt = \int_{0}^{\pi/6} \frac{1 + \cos(2t)}{2} \, dt
$$

$$
= \left( \frac{t}{2} + \frac{\sin(2t)}{4} \right)|_{t=0}^{\pi/6} = \frac{\pi}{12} + \frac{\sqrt{3}}{8}.
$$

**Bonus Problem 5 (15 pts.)** Prove that the sequence \( \{y_n\} \) is bounded, where

$$
y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n, \quad n = 1, 2, 3, \ldots
$$

The function $h(x) = \frac{1}{x}$ is integrable on every interval $J = [a, b] \subset (0, \infty)$. Hence for any partition $P$ of the interval $J$ the lower Darboux sum $L(h, P)$ and the upper Darboux sum $U(h, P)$ satisfy

$$
L(h, P) \leq \int_{a}^{b} h(x) \, dx \leq U(h, P).
$$

In the case $J = [1, n]$, where $n > 1$ is an integer, and $P = \{1, 2, \ldots, n\}$ we obtain

$$
L(h, P) = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad \int_{1}^{n} h(x) \, dx = \log n, \quad U(h, P) = 1 + \frac{1}{2} + \cdots + \frac{1}{n - 1}.
$$

Then the above inequalities imply that $1/n \leq y_n \leq 1$. Thus the sequence \( \{y_n\} \) is bounded.