Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

**Problem 1 (20 pts.)** Prove the Chain Rule: if a function $f$ is differentiable at a point $c$ and a function $g$ is differentiable at $f(c)$, then the composition $g \circ f$ is differentiable at $c$ and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Since the function $f$ is differentiable at the point $c$, the domain of $f$ contains an open interval $I_0 = (c - \delta_0, c + \delta_0)$ for some $\delta_0 > 0$. Since $g$ is differentiable at $f(c)$, the domain of $g$ contains an open interval $J = (f(c) - \varepsilon_0, f(c) + \varepsilon_0)$ for some $\varepsilon_0 > 0$. The differentiability of $f$ at $c$ implies that $f$ is continuous at that point. Hence there exists $\delta_1 \in (0, \delta_0)$ such that $|f(x) - f(c)| < \varepsilon_0$ whenever $|x - c| < \delta_1$. Then the composition $g \circ f$ is well defined on the interval $I_1 = (c - \delta_1, c + \delta_1)$. Consider a set $E = \{x \in I_1 \mid f(x) \neq f(c)\}$. For any $x \in E$,

$$\frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}.$$ 

As $x \to c$ within the subset $E$, we have $f(x) \to f(c)$ while $f(x) \neq f(c)$. Therefore

$$\lim_{x \to c_{x \in E}} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} = \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)).$$

Consequently,

$$\lim_{x \to c_{x \in E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = g'(f(c)) \cdot f'(c).$$

In the case $f(x) \neq f(c)$ for all $x$ in a sufficiently small punctured neighborhood of $c$, the restriction $x \in E$ in the above limit is redundant and we are done. Otherwise we also need to consider the limit as $x \to c$ within the complement of $E$. Notice that $g(f(x)) - g(f(c)) = f(x) - f(c) = 0$ for all $x \notin E$. Hence

$$\lim_{x \to c_{x \notin E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = \lim_{x \to c_{x \notin E}} \frac{f(x) - f(c)}{x - c} = 0.$$

In particular, we have $f'(c) = 0$ in this case so that

$$\lim_{x \to c_{x \notin E}} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} = g'(f(c)) \cdot f'(c).$$

**Problem 2 (25 pts.)** Find the following limits of functions:

(i) $\lim_{x \to 0} (1 + x)^{1/x}$,  
(ii) $\lim_{x \to +\infty} (1 + x)^{1/x}$,  
(iii) $\lim_{x \to 0^+} x^x$.  

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The function \( f(x) = (1 + x)^{1/x} \) is well defined on \((-1, 0) \cup (0, \infty)\). Since \( f(x) > 0 \) for all \( x > -1, x \neq 0 \), a function \( g(x) = \log f(x) \) is well defined on \((-1, 0) \cup (0, \infty)\) as well. For any \( x > -1, x \neq 0 \), we have \( g(x) = \log(1 + x)^{1/x} = x^{-1} \log(1 + x) \). Hence \( g = h_1/h_2 \), where the functions \( h_1(x) = \log(1 + x) \) and \( h_2(x) = x \) are continuously differentiable on \((-1, \infty)\). Since \( h_1(0) = h_2(0) = 0 \), it follows that \( \lim_{x \to 0} h_1(x) = \lim_{x \to 0} h_2(x) = 0 \). By l’Hôpital’s Rule,

\[
\lim_{x \to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x \to 0} \frac{h_1'(x)}{h_2'(x)}
\]

assuming the latter limit exists. Since \( h'_1(0) = (1 + x)^{-1}|_{x=0} = 1 \) and \( h'_2(0) = 1 \), we obtain

\[
\lim_{x \to 0} \frac{h_1(x)}{h_2(x)} = \lim_{x \to 0} \frac{h_1'(x)}{h_2'(x)} = \frac{0}{1} = 0.
\]

Further, \( \lim_{x \to +\infty} h_1(x) = \lim_{x \to +\infty} h_2(x) = +\infty \). At the same time, \( h_1'(x) = (1 + x)^{-1} \to 0 \) as \( x \to +\infty \) while \( h_2' \) is identically 1. Using l’Hôpital’s Rule and a limit theorem, we obtain

\[
\lim_{x \to +\infty} \frac{h_1(x)}{h_2(x)} = \lim_{x \to +\infty} \frac{h_1'(x)}{h_2'(x)} = \lim_{x \to +\infty} \frac{h_1'(x)}{h_2'(x)} = 0 = 1.
\]

Since \( f = e^g \), a composition of \( g \) with a continuous function, it follows that

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} e^g(x) = \exp \left( \lim_{x \to 0} g(x) \right) = e^1 = e, \quad \lim_{x \to +\infty} f(x) = \exp \left( \lim_{x \to +\infty} g(x) \right) = e^0 = 1.
\]

Now let us consider a function \( F(x) = x^x, x > 0 \). Since \( F \) takes positive values, a function \( G(x) = \log F(x) \) is well defined on \((0, \infty)\). We have \( G(x) = \log(x^x) = x \log x \) for all \( x > 0 \) so that \( G = H_1/H_2 \), where \( H_1(x) = \log x \) and \( H_2(x) = x^{-1} \) are differentiable functions on \((0, \infty)\). We observe that \( \lim_{x \to 0^+} H_1(x) = -\infty \) and \( \lim_{x \to 0^+} H_2(x) = +\infty \). By l’Hôpital’s Rule,

\[
\lim_{x \to 0^+} \frac{H_1(x)}{H_2(x)} = \lim_{x \to 0^+} \frac{H_1'(x)}{H_2'(x)} = \lim_{x \to 0^+} \frac{(\log x)'}{x^{-1}'} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} (-x) = 0.
\]

Consequently,

\[
\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} e^{G(x)} = \exp \left( \lim_{x \to 0^+} G(x) \right) = e^0 = 1.
\]

**Problem 3 (20 pts.)** Find the limit of a sequence

\[
x_n = \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad n = 1, 2, \ldots,
\]

where \( k \) is a natural number.

The general element of the sequence can be represented as

\[
x_n = \frac{1^k + 2^k + \cdots + n^k}{n^k} \cdot \frac{1}{n} = \left( \frac{1}{n} \right)^k \frac{1}{n} + \left( \frac{2}{n} \right)^k \frac{1}{n} + \cdots + \left( \frac{n}{n} \right)^k \frac{1}{n},
\]
which shows that $x_n$ is a Riemann sum of the function $f(x) = x^k$ on the interval $[0, 1]$ that corresponds to the partition $P_n = \{0, 1/n, 2/n, \ldots, (n - 1)/n, 1\}$ and samples $t_j = j/n$, $j = 1, 2, \ldots, n$. The norm of the partition is $\|P_n\| = 1/n$. Since $\|P_n\| \to 0$ as $n \to \infty$ and the function $f$ is integrable on $[0,1]$, the Riemann sums $x_n$ converge to the integral:

$$\lim_{n \to \infty} x_n = \int_0^1 x^k \, dx = \frac{x^{k+1}}{k+1} \bigg|_{x=0}^1 = \frac{1}{k+1}.$$

**Problem 4 (25 pts.)** Find indefinite integrals and evaluate definite integrals:

(i) $\int \frac{x^2}{1-x} \, dx,$  \hspace{0.5cm} (ii) $\int_0^\pi \sin^2(2x) \, dx,$  \hspace{0.5cm} (iii) $\int \log^3 x \, dx,$

(iv) $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, dx,$  \hspace{0.5cm} (v) $\int_0^1 \frac{1}{\sqrt{4-x^2}} \, dx.$

To find the indefinite integral of a rational function $y(x) = x^2/(1 - x)$, we expand it into the sum of a polynomial and a simple fraction:

$$\frac{x^2}{1-x} = \frac{x^2 - 1 + 1}{1-x} = \frac{x^2 - 1}{1-x} + \frac{1}{1-x} = \frac{(x-1)(x+1)}{1-x} + \frac{1}{1-x} = -x - 1 + \frac{1}{x-1}.$$

Since the domain of the function $y$ is $(-\infty, 1) \cup (1, \infty)$, the indefinite integral has different representations on the intervals $(-\infty, 1)$ and $(1, \infty)$:

$$\int \frac{x^2}{1-x} \, dx = \int \left( -x - 1 \right) \frac{1}{1-x} \, dx = \begin{cases} -x^2/2 - x - \log(1-x) + C_1, & x < 1, \\ -x^2/2 - x - \log(x-1) + C_2, & x > 1. \end{cases}$$

To integrate the function $y(x) = \sin^2(2x)$, we use a trigonometric formula $1 - \cos(2\alpha) = 2\sin^2 \alpha$ and a new variable $u = 4x$:

$$\int_0^\pi \sin^2(2x) \, dx = \int_0^\pi \frac{1 - \cos(4x)}{2} \, dx = \int_0^\pi \frac{1 - \cos(4x)}{8} \, d(4x)$$

$$= \int_0^{4\pi} \frac{1 - \cos u}{8} \, du = \frac{u - \sin u}{8} \bigg|_{u=0}^{4\pi} = \frac{\pi}{2}.$$

To find the antiderivative of the function $y(x) = \log^3 x$, we integrate by parts three times:

$$\int \log^3 x \, dx = x \log^3 x - \int x \, d(\log^3 x) = x \log^3 x - \int x \, (3 \log^2 x) \, dx = x \log^3 x - \int 3 \log^2 x \, dx$$

$$= x \log^3 x - 3x \log^2 x + \int x \, d(3 \log^2 x) = x \log^3 x - 3x \log^2 x + \int x \, (6 \log x) \, dx$$

$$= x \log^3 x - 3x \log^2 x + 6x \log x - \int x \, d(6 \log x)$$

$$= x \log^3 x - 3x \log^2 x + 6x \log x - \int (6 \log x)' \, dx = x \log^3 x - 3x \log^2 x + 6x \log x - \int 6 \, dx.$$
Similarly, we prove that the set $p \in \mathbb{R}$ changes from 0 to 1 when $c$ sets $E \subset $ Show that a function $\log$ denotes a polynomial and $d$ is a polynomial. Hence $\frac{d}{2} = \int_0^{\pi/6} \frac{1}{\sqrt{4 - x^2}} \, dx$. To integrate the function $y(x) = 1/\sqrt{4 - x^2}$, we use a substitution $x = 2\sin t$ (observe that $x$ changes from 0 to 1 when $t$ changes from 0 to $\pi/6$):

$$
\int_0^{\pi/6} \frac{1}{\sqrt{4 - x^2}} \, dx = \int_0^{\pi/6} \frac{1}{\sqrt{4 - (2\sin t)^2}} \, d(2\sin t) = \int_0^{\pi/6} \frac{(2\sin t)'}{\sqrt{4 - 4\sin^2 t}} \, dt
$$

$$
= \int_0^{\pi/6} \frac{2\cos t}{\sqrt{4\cos^2 t}} \, dt = \int_0^{\pi/6} \frac{2\cos t}{2\cos t} \, dt = \int_0^{\pi/6} 1 \, dx = \frac{\pi}{6}.
$$

**Bonus Problem 5 (15 pts.)** Suppose that a function $p : \mathbb{R} \rightarrow \mathbb{R}$ is locally a polynomial, which means that for every $c \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $p$ coincides with a polynomial on the interval $(c - \varepsilon, c + \varepsilon)$. Prove that $p$ is a polynomial.

For any $c \in \mathbb{R}$ let $p_c$ denote a polynomial and $\varepsilon_c$ denote a positive number such that $p(x) = p_c(x)$ for all $x \in (c - \varepsilon_c, c + \varepsilon_c)$. We are going to show that $p = p_0$ on the entire real line. Consider two sets $E_+ = \{ x > 0 \mid p(x) \neq p_0(x) \}$ and $E_- = \{ x < 0 \mid p(x) \neq p_0(x) \}$. Assume that the set $E_+$ is not empty. Clearly, $E_+$ is bounded below. Hence $d = \inf E_+$ is a well-defined real number. Note that $E_+ \subset [\varepsilon_0, \infty)$. Therefore $d \geq \varepsilon_0 > 0$. Observe that $p(x) = p_0(x)$ for $x \in (0, d)$ and $p(x) = p_d(x)$ for $x \in (d - \varepsilon_d, d + \varepsilon_d)$. The interval $(0, d)$ overlaps with the interval $(d - \varepsilon_d, d + \varepsilon_d)$. Hence $p_d$ coincides with $p_0$ on the intersection $(0, d) \cap (d - \varepsilon_d, d + \varepsilon_d)$. Equivalently, the difference $p_d - p_0$ is zero on $(0, d) \cap (d - \varepsilon_d, d + \varepsilon_d)$. Since $p_d - p_0$ is a polynomial and any nonzero polynomial has only finitely many roots, we conclude that $p_d - p_0$ is identically 0. Then the polynomials $p_d$ and $p_0$ are the same. It follows that $p(x) = p_0(x)$ for $x \in (0, d + \varepsilon_d)$ so that $d \neq \inf E_+$, a contradiction. Thus $E_+ = \emptyset$.

Similarly, we prove that the set $E_-$ is empty as well. Since $E_+ = E_- = \emptyset$, the function $p$ coincides with the polynomial $p_0$.

**Bonus Problem 6 (15 pts.)** Show that a function

$$
f(x) = \begin{cases} 
\exp \left( -\frac{1}{1-x^2} \right) & \text{if } |x| < 1, \\
0 & \text{if } |x| \geq 1
\end{cases}
$$

is infinitely differentiable on $\mathbb{R}$.

Consider a function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = e^{-1/x}$ for $x > 0$ and $h(x) = 0$ for $x \leq 0$. It is easy to verify that $f(x) = h(2 + 2x)h(2 - 2x)$ for all $x \in \mathbb{R}$. Since the set $C^\infty(\mathbb{R})$ of infinitely
differentiable functions is closed under multiplication and composition of functions, it is enough to prove that \( h \in C^\infty(\mathbb{R}) \).

Obviously, the function \( h \) is infinitely differentiable on \(( -\infty, 0 )\) and on \(( 0, \infty )\). Moreover, all derivatives on \(( -\infty, 0 )\) are identically zero. Let us prove that for any integer \( n \geq 0 \) there exists a polynomial \( p_n \) such that \( h^{(n)}(x) = p_n(x)x^{-2n}e^{-1/x} \) for all \( x > 0 \), where \( h^{(n)} \) is the \( n \)-th derivative of \( h \) for \( n > 0 \) and \( h^{(0)} = h \). The proof is by induction on \( n \). The base case \( n = 0 \) is trivial, with \( p_0 = 1 \).

Now assume that the above representation holds for some integer \( n \geq 0 \). Then

\[
\begin{align*}
h^{(n+1)}(x) &= \left( h^{(n)}(x) \right)' = \left( p_n(x)x^{-2n}e^{-1/x} \right)' \\
&= p'_n(x)x^{-2n}e^{-1/x} + p_n(x)(-2n)x^{-2n-1}e^{-1/x} + p_n(x)x^{-2n}e^{-1/x}x^{-2} \\
&= p'_n(x)x^{-2n}e^{-1/x} + p_n(x)(-2n)x^{-2n-1}e^{-1/x} + p_n(x)x^{-2n}e^{-1/x}x^{-2} \\
&= p_{n+1}(x)x^{-2(n+1)}e^{-1/x},
\end{align*}
\]

where \( p_{n+1}(x) = p'_n(x)x^2 - 2np_n(x)x + p_n(x) \) is a polynomial. Since \( t^k/e^t \to 0 \) as \( t \to +\infty \) for any \( k > 0 \) and \( 1/x \to +\infty \) as \( x \to 0^+ \), we obtain that \( x^{-k}e^{-1/x} \to 0 \) as \( x \to 0^+ \) for any \( k > 0 \). Then it follows from the above that \( h^{(n)}(x)/x \to 0 \) as \( x \to 0^+ \) for any \( n \geq 0 \). Now it easily follows by induction on \( n \in \mathbb{N} \) that the function \( h \) is \( n \) times differentiable at 0 and \( h^{(n)}(0) = 0 \). Thus \( h \in C^\infty(\mathbb{R}) \).