Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

**Problem 1 (15 pts.)** Find a quadratic polynomial \( p(x) \) such that \( p(-1) = p(3) = 6 \) and \( p'(2) = p(1) \).

Let \( p(x) = a + bx + cx^2 \). Then \( p(-1) = a - b + c \), \( p(1) = a + b + c \), and \( p(3) = a + 3b + 9c \). Also, \( p'(x) = b + 2cx \) so that \( p'(2) = b + 4c \). The coefficients \( a, b, \) and \( c \) are to be chosen so that

\[
\begin{align*}
  &\begin{cases}
    a - b + c = 6, \\
    a + 3b + 9c = 6, \\
    b + 4c = a + b + c
  \end{cases} \\
  \iff &\begin{cases}
    a - b + c = 6, \\
    a + 3b + 9c = 6, \\
    a - 3c = 0.
  \end{cases}
\end{align*}
\]

This is a system of linear equations. To solve it, we convert the augmented matrix to reduced row echelon form using elementary row operations:

\[
\begin{pmatrix}
  1 & -1 & 1 & 6 \\
  1 & 3 & 9 & 6 \\
  1 & 0 & -3 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -3 & 0 \\
  0 & -1 & 4 & 6 \\
  0 & 3 & 12 & 6
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -3 & 0 \\
  0 & 1 & 4 & 6 \\
  0 & 0 & 24 & 24
\end{pmatrix} \rightarrow \begin{pmatrix}
  1 & 0 & -3 & 0 \\
  0 & 1 & 4 & 6 \\
  0 & 0 & 1 & 1
\end{pmatrix}
\]

We obtain that the system has a unique solution: \( a = 3, b = -2, \) and \( c = 1 \). Thus \( p(x) = x^2 - 2x + 3 \).

**Problem 2 (20 pts.)** Consider a linear transformation \( L : \mathbb{R}^5 \to \mathbb{R}^2 \) given by

\[
L(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_3 + x_5, 2x_1 - x_2 + x_4).
\]

Find a basis for the null-space of \( L \), then extend it to a basis for \( \mathbb{R}^5 \).

The null-space \( \mathcal{N}(L) \) consists of all vectors \( \mathbf{x} \in \mathbb{R}^5 \) such that \( L(\mathbf{x}) = \mathbf{0} \). This is the solution set of the following systems of linear equations:

\[
\begin{align*}
  &\begin{cases}
    x_1 + x_3 + x_5 = 0 \\
    2x_1 - x_2 + x_4 = 0
  \end{cases} \iff
  &\begin{cases}
    x_1 + x_3 + x_5 = 0 \\
    -x_2 - 2x_3 + x_4 - 2x_5 = 0
  \end{cases} \\
  \iff &\begin{cases}
    x_1 + x_3 + x_5 = 0 \\
    x_2 + 2x_3 - x_4 + 2x_5 = 0
  \end{cases} \iff
  &\begin{cases}
    x_1 = -x_3 - x_5 \\
    x_2 = -2x_3 + x_4 - 2x_5
  \end{cases}
\end{align*}
\]

The general solution of the system is

\[
\mathbf{x} = (-t_1 - t_3, -2t_1 + t_2 - 2t_3, t_1, t_2, t_3) = t_1(-1, -2, 1, 0, 0) + t_2(0, 1, 0, 1, 0) + t_3(-1, -2, 0, 0, 1),
\]

where \( t_1, t_2, t_3 \) are any scalars. This is the solution set of the system of linear equations determined by \( L \).
where \( t_1, t_2, t_3 \) are arbitrary real numbers. We obtain that the null-space \( \mathcal{N}(L) \) is spanned by vectors \( v_1 = (-1, -2, 1, 0, 0), \ v_2 = (0, 1, 0, 1, 0), \) and \( v_3 = (-1, -2, 0, 0, 1). \) The last three coordinates of these vectors form the standard basis for \( \mathbb{R}^3. \) It follows that the vectors \( v_1, v_2, v_3 \) are linearly independent. Hence they form a basis for \( \mathcal{N}(L). \)

To extend the basis for \( \mathcal{N}(L) \) to a basis for \( \mathbb{R}^5, \) we need two more vectors. We can use two vectors from the standard basis. For example, the vectors \( v_1, v_2, v_3, e_1, e_2 \) form a basis for \( \mathbb{R}^5. \) To verify this, we show that a \( 5 \times 5 \) matrix with these vectors as columns has a nonzero determinant:

\[
\begin{vmatrix}
-1 & 0 & -1 & 1 & 0 \\
-2 & 1 & -2 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{vmatrix}
= 1.
\]

**Problem 3 (20 pts.)** Let \( v_1 = (1, 1, 1), v_2 = (1, 1, 0), \) and \( v_3 = (1, 0, 1). \) Let \( T: \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear operator on \( \mathbb{R}^3 \) such that \( T(v_1) = v_2, T(v_2) = v_3, T(v_3) = v_1. \) Find the matrix of the operator \( T \) relative to the standard basis.

Let \( U \) be a \( 3 \times 3 \) matrix such that its columns are vectors \( v_1, v_2, v_3: \)

\[
U = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}.
\]

To determine whether \( v_1, v_2, v_3 \) is a basis for \( \mathbb{R}^3, \) we find the determinant of \( U: \)

\[
\det U = \begin{vmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{vmatrix}
= 1
\]

Since \( \det U \neq 0, \) the vectors \( v_1, v_2, v_3 \) are linearly independent. Therefore they form a basis for \( \mathbb{R}^3. \) It follows that the operator \( T \) is defined well and uniquely.

The matrix of the operator \( T \) relative to the basis \( v_1, v_2, v_3 \) is

\[
B = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

Since the matrix \( U \) is the transition matrix from \( v_1, v_2, v_3 \) to the standard basis, the matrix of \( T \) relative to the standard basis is \( A = UBU^{-1}. \)

To find the inverse \( U^{-1}, \) we merge the matrix \( U \) with the identity matrix \( I \) into one \( 3 \times 6 \) matrix and apply row reduction to convert the left half \( U \) of this matrix into \( I. \) Simultaneously, the right half \( I \) will be converted into \( U^{-1}: \)

\[
(U|I) = \begin{vmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1
\end{vmatrix}
\]

\[
= \begin{vmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1
\end{vmatrix}
\]

\[
= \begin{vmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 1 & 0
\end{vmatrix}
\]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
-1 & 1 & 1 \\
1 & 0 & -1 \\
1 & -1 & 0
\end{pmatrix} = (I|U^{-1}).
\]

Thus

\[
A = UBU^{-1} = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
-1 & 1 & 1 \\
1 & 0 & -1 \\
1 & -1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
-1 & 1 & 1 \\
1 & 0 & -1 \\
1 & -1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
2 & -1 & -1
\end{pmatrix}.
\]

**Problem 4 (20 pts.)** Let \( R : \mathbb{R}^3 \to \mathbb{R}^3 \) be the operator of orthogonal reflection in the plane \( \Pi \) spanned by vectors \( u_1 = (1, 0, -1) \) and \( u_2 = (1, -1, 3) \). Find the image of the vector \( u = (2, 3, 4) \) under this operator.

By definition of the orthogonal reflection, \( R(x) = x \) for any vector \( x \in \Pi \) and \( R(y) = -y \) for any vector \( y \) orthogonal to the plane \( \Pi \). The vector \( u \) is uniquely decomposed as \( u = p + o \), where \( p \in \Pi \) and \( o \in \Pi^\perp \). Then \( R(u) = R(p + o) = R(p) + R(o) = p - o \).

The component \( p \) is the orthogonal projection of the vector \( u \) onto the plane \( \Pi \). We can compute it using the formula

\[
p = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2,
\]

in which \( v_1, v_2 \) is an arbitrary orthogonal basis for \( \Pi \). To get such a basis, we apply the Gram-Schmidt process to the basis \( u_1, u_2 \):

\[
\begin{align*}
v_1 &= u_1 = (1, 0, -1), \\
v_2 &= u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (1, -1, 3) - \frac{-2}{2} (1, 0, -1) = (2, -1, 2).
\end{align*}
\]

Now \( p = -\frac{2}{2} (1, 0, -1) + \frac{9}{9} (2, -1, 2) = (1, -1, 3) \).

Then \( o = u - p = (1, 4, 1) \). Finally, \( R(u) = p - o = (0, -5, 2) \).

**Problem 5 (25 pts.)** Consider the vector space \( W \) of all polynomials of degree at most 3 in variables \( x \) and \( y \) with real coefficients. Let \( D \) be a linear operator on \( W \) given by \( D(p) = \frac{\partial p}{\partial x} \) for any \( p \in W \). Find the Jordan canonical form of the operator \( D \).

The vector space \( W \) is 10-dimensional. It has a basis of monomials: \( 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3 \).

Note that \( D(x^m y^k) = mx^{m-1} y^k \) if \( m > 0 \) and \( D(x^m y^k) = 0 \) otherwise. It follows that the operator \( D^4 \) maps each monomial to zero, which implies that this operator is identically zero. As a consequence, \( 0 \) is the only eigenvalue of the operator \( D \).

To determine the Jordan canonical form of \( D \), we need to determine the null-spaces of its iterations. Indeed, \( \dim \mathcal{N}(D) \) is the total number of Jordan blocks in the Jordan canonical form of \( D \). Next, \( \dim \mathcal{N}(D^2) - \dim \mathcal{N}(D) \) is the number of Jordan blocks of dimensions at least 2 \( \times \) 2. Further, \( \dim \mathcal{N}(D^3) - \dim \mathcal{N}(D^2) \) is the number of Jordan blocks of dimensions at least 3 \( \times \) 3, and so on...
The null-space $\mathcal{N}(D)$ is 4-dimensional, it is spanned by $1, y, y^2, y^3$. The null-space $\mathcal{N}(D^2)$ is 7-dimensional, it is spanned by $1, y, y^2, y^3, x, xy, xy^2$. The null-space $\mathcal{N}(D^3)$ is 9-dimensional, it is spanned by $1, y, y^2, y^3, x, xy, xy^2, x^2, x^2y$. The null-space $\mathcal{N}(D^4)$ is the entire 10-dimensional space $W$. It follows that the Jordan canonical form of $D$ contains one Jordan block of dimensions $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

**Bonus Problem 6 (15 pts.)** An upper triangular matrix is called unipotent if all diagonal entries are equal to 1. Prove that the inverse of a unipotent matrix is also unipotent.

Let $\mathcal{U}$ denote the class of elementary row operations that add a scalar multiple of row $#i$ to row $#j$, where $i$ and $j$ satisfy $j < i$. It is easy to see that such an operation transforms a unipotent matrix into another unipotent matrix.

It remains to observe that any unipotent matrix $A$ (which is in row echelon form) can be converted into the identity matrix $I$ (which is its reduced row echelon form) by applying only operations from the class $\mathcal{U}$. Now the same sequence of elementary row operations converts $I$ into the inverse matrix $A^{-1}$. Since the identity matrix is unipotent, so is $A^{-1}$. 