Lecture 7:
Linear transformations.
Range and null-space.
Definition. Given vector spaces $V_1$ and $V_2$, a mapping $L: V_1 \to V_2$ is linear (or $\mathbb{F}$-linear) if

$$L(x + y) = L(x) + L(y),$$

$$L(rx) = rL(x)$$

for any $x, y \in V_1$ and $r \in \mathbb{F}$.

A linear mapping $\ell: V \to \mathbb{F}$ is called a linear functional on $V$.

If $V_1 = V_2$ (or if both $V_1$ and $V_2$ are functional spaces) then a linear mapping $L: V_1 \to V_2$ is called a linear operator.
**Definition.** Given vector spaces $V_1$ and $V_2$, a mapping $L : V_1 \rightarrow V_2$ is linear (or $\mathbb{F}$-linear) if
\[
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\]
\[
L(rx) = rL(x)
\]
for any $x, y \in V_1$ and $r \in \mathbb{F}$.

**Remark.** A function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = ax + b$ is a linear transformation of the vector space $\mathbb{R}$ only if $b = 0$. 
Basic properties of linear mappings

Let $L : V_1 \rightarrow V_2$ be a linear mapping.

- $L(r_1v_1 + \cdots + r_kv_k) = r_1L(v_1) + \cdots + r_kL(v_k)$ for all $k \geq 1$, $v_1, \ldots, v_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{F}$.

$L(r_1v_1 + r_2v_2) = L(r_1v_1) + L(r_2v_2) = r_1L(v_1) + r_2L(v_2),$
$L(r_1v_1 + r_2v_2 + r_3v_3) = L(r_1v_1 + r_2v_2) + L(r_3v_3) = r_1L(v_1) + r_2L(v_2) + r_3L(v_3),$ and so on.

- $L(0_1) = 0_2$, where $0_1$ and $0_2$ are zero vectors in $V_1$ and $V_2$, respectively.

$L(0_1) = L(00_1) = 0L(0_1) = 0_2.$

- $L(-v) = -L(v)$ for any $v \in V_1$.

$L(-v) = L((-1)v) = (-1)L(v) = -L(v).$
Examples of linear mappings

- **Scaling** $L : V \to V$, $L(v) = sv$, where $s \in \mathbb{F}$.
  
  $L(x + y) = s(x + y) = sx + sy = L(x) + L(y)$,
  
  $L(rx) = s(rx) = (sr)x = r(sx) = rL(x)$.

- **Dot product with a fixed vector**
  
  $\ell : \mathbb{R}^n \to \mathbb{R}$, $\ell(v) = v \cdot v_0$, where $v_0 \in \mathbb{R}^n$.
  
  $\ell(x + y) = (x + y) \cdot v_0 = x \cdot v_0 + y \cdot v_0 = \ell(x) + \ell(y)$,
  
  $\ell(rx) = (rx) \cdot v_0 = r(x \cdot v_0) = r\ell(x)$.

- **Cross product with a fixed vector**
  
  $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(v) = v \times v_0$, where $v_0 \in \mathbb{R}^3$.

- **Multiplication by a fixed matrix**
  
  $L : \mathbb{R}^n \to \mathbb{R}^m$, $L(v) = Av$, where $A$ is an $m \times n$ matrix and all vectors are column vectors.
Linear mappings of functional vector spaces

- **Evaluation at a fixed point**
  \[ \ell : \mathcal{F}(S) \rightarrow \mathbb{R}, \quad \ell(f) = f(a), \text{ where } a \in S. \]

- **Multiplication by a fixed function**
  \[ L : \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R}), \quad L(f) = gf, \text{ where } g \in \mathcal{F}(\mathbb{R}). \]

- **Differentiation**
  \[ D : C^1(\mathbb{R}) \rightarrow C(\mathbb{R}), \quad L(f) = f'. \]
  \[ D(f + g) = (f + g)' = f' + g' = D(f) + D(g), \]
  \[ D(rf) = (rf)' = rf' = rD(f). \]

- **Integration over a finite interval**
  \[ \ell : C(\mathbb{R}) \rightarrow \mathbb{R}, \quad \ell(f) = \int_{a}^{b} f(x) \, dx, \text{ where } a, b \in \mathbb{R}, \quad a < b. \]
\( \mathcal{M}_{m,n}(\mathbb{R}) \): the space of \( m \times n \) matrices.

- \( \alpha : \mathcal{M}_{m,n}(\mathbb{R}) \to \mathcal{M}_{n,m}(\mathbb{R}), \ \alpha(A) = A^t \), transpose of \( A \).

\[
\alpha(A + B) = \alpha(A) + \alpha(B) \iff (A + B)^t = A^t + B^t.
\]

\[
\alpha(rA) = r \alpha(A) \iff (rA)^t = rA^t.
\]

Hence \( \alpha \) is linear.

- \( \beta : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}, \ \beta(A) = \det A \).

Let \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

Then \( A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

We have \( \det(A) = \det(B) = 0 \) while \( \det(A + B) = 1 \).

Hence \( \beta(A + B) \neq \beta(A) + \beta(B) \) so that \( \beta \) is not linear.
More properties of linear mappings

- If a linear mapping $L: V \to W$ is invertible then the inverse mapping $L^{-1}: W \to V$ is also linear.

Given vectors $w_1, w_2 \in W$, let $v_1 = L^{-1}(w_1), v_2 = L^{-1}(w_2)$. Since $L$ is linear, $L(v_1 + v_2) = L(v_1) + L(v_2) = w_1 + w_2$. That is, $L^{-1}(w_1 + w_2) = v_1 + v_2 = L^{-1}(w_1) + L^{-1}(w_2)$.

Given a vector $w \in W$, let $v = L^{-1}(w)$. Since $L$ is linear, for any scalar $r$ we have $L(rv) = rL(v) = rw$. That is, $L^{-1}(rw) = rv = rL^{-1}(w)$.

- If $L: V \to W$ and $M: W \to X$ are linear mappings then the composition $M \circ L: V \to X$ is also linear.

$$(M \circ L)(v_1 + v_2) = M(L(v_1 + v_2)) = M(L(v_1) + L(v_2)) = M(L(v_1)) + M(L(v_2)) = (M \circ L)(v_1) + (M \circ L)(v_2).$$

$$(M \circ L)(rv) = M(L(rv)) = M(rL(v)) = rM(L(v)).$$
Let $W$ be a vector space over a field $\mathbb{F}$. For any nonempty set $S$ let $\mathcal{F}(S, W)$ denote the set of all mappings $f : S \to W$. The set $\mathcal{F}(S, W)$ is naturally endowed with the structure of a vector space over $\mathbb{F}$ (this was already done before in the case $W = \mathbb{R}$). Namely, for any functions $f, g \in \mathcal{F}(S, W)$ we define the sum $f + g$ by $(f + g)(x) = f(x) + g(x)$, $x \in S$. For any function $f \in \mathcal{F}(S, W)$ and scalar $r \in \mathbb{F}$ we define the scalar multiple $rf$ by $(rf)(x) = r \cdot f(x)$, $x \in S$.

For any vector space $V$ over $\mathbb{F}$ we denote by $\mathcal{L}(V, W)$ a subset of $\mathcal{F}(V, W)$ consisting of all linear transformations from $V$ to $W$.

**Theorem** $\mathcal{L}(V, W)$ is a subspace of $\mathcal{F}(V, W)$. 
Examples of linear differential operators

- an ordinary differential operator

\[ L : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2, \]

where \( g_0, g_1, g_2 \) are smooth functions on \( \mathbb{R} \).

That is, \( L(f) = g_0 f'' + g_1 f' + g_2 f \).

- Laplace's operator \( \Delta : C^\infty(\mathbb{R}^2) \rightarrow C^\infty(\mathbb{R}^2) \),

\[ \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \]

(a.k.a. the Laplacian; also denoted by \( \nabla^2 \)).
Range and null-space

Let $V, W$ be vector spaces and $L : V \to W$ be a linear mapping.

**Definition.** The **range** (or **image**) of $L$ is the set of all vectors $w \in W$ such that $w = L(v)$ for some $v \in V$. The range of $L$ is denoted $\mathcal{R}(L)$ (or $L(V)$).

The **null-space** (or **kernel**) of $L$, denoted $\mathcal{N}(L)$, is the set of all vectors $v \in V$ such that $L(v) = 0$.

**Theorem (i)** The range $\mathcal{R}(L)$ is a subspace of $W$.

**(ii)** The null-space $\mathcal{N}(L)$ is a subspace of $V$.

$\dim \mathcal{R}(L)$ is called the **rank** of the transformation $L$.

$\dim \mathcal{N}(L)$ is called the **nullity** of $L$. 
**Dimension Theorem**

**Theorem**  Let \( L : V \rightarrow W \) be a linear mapping of a finite-dimensional vector space \( V \) to a vector space \( W \). Then \( \dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V \).

The null-space \( \mathcal{N}(L) \) is a subspace of \( V \). It is finite-dimensional since the vector space \( V \) is.

Take a basis \( v_1, v_2, \ldots, v_k \) for the subspace \( \mathcal{N}(L) \), then extend it to a basis \( v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_m \) for the entire space \( V \).

**Claim**  Vectors \( L(u_1), L(u_2), \ldots, L(u_m) \) form a basis for the range of \( L \).

Assuming the claim is proved, we obtain
\[
\dim \mathcal{R}(L) = m, \quad \dim \mathcal{N}(L) = k, \quad \dim V = k + m.
\]
Claim  Vectors \( L(u_1), L(u_2), \ldots, L(u_m) \) form a basis for the range of \( L \).

Proof (spanning):  Any vector \( w \in \mathcal{R}(L) \) is represented as \( w = L(v) \), where \( v \in V \). Then

\[
v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m
\]

for some \( \alpha_i, \beta_j \in F \). It follows that

\[
w = L(v) = \alpha_1 L(v_1) + \cdots + \alpha_k L(v_k) + \beta_1 L(u_1) + \cdots + \beta_m L(u_m)
\]

\[
= \beta_1 L(u_1) + \cdots + \beta_m L(u_m).
\]

Note that \( L(v_i) = 0 \) since \( v_i \in \mathcal{N}(L) \).
Thus \( \mathcal{R}(L) \) is spanned by the vectors \( L(u_1), \ldots, L(u_m) \).
Claim  Vectors  $L(u_1), L(u_2), \ldots, L(u_m)$ form a basis for the range of $L$.

Proof (linear independence):  Assume that 

$$t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0$$

for some $t_i \in \mathbb{F}$. Let $u = t_1 u_1 + t_2 u_2 + \cdots + t_m u_m$. Since 

$$L(u) = t_1 L(u_1) + t_2 L(u_2) + \cdots + t_m L(u_m) = 0,$$

the vector $u$ belongs to the null-space of $L$. Therefore $u = s_1 v_1 + s_2 v_2 + \cdots + s_k v_k$ for some $s_j \in \mathbb{F}$. It follows that 

$$t_1 u_1 + t_2 u_2 + \cdots + t_m u_m - s_1 v_1 - s_2 v_2 - \cdots - s_k v_k = u - u = 0.$$

Linear independence of vectors $v_1, \ldots, v_k, u_1, \ldots, u_m$ implies that $t_1 = \cdots = t_m = 0$ (as well as $s_1 = \cdots = s_k = 0$). Thus the vectors $L(u_1), L(u_2), \ldots, L(u_m)$ are linearly independent.