MATH 423
Linear Algebra II

Lecture 11:
Change of coordinates (continued).
Isomorphism of vector spaces.
Change of coordinates

Let $V$ be a vector space of dimension $n$.
Let $v_1, v_2, \ldots, v_n$ be a basis for $V$ and $g_1 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.
Let $u_1, u_2, \ldots, u_n$ be another basis for $V$ and $g_2 : V \to \mathbb{F}^n$ be the coordinate mapping corresponding to this basis.

The composition $g_2 \circ g_1^{-1}$ is a linear operator on $\mathbb{F}^n$. It has the form $x \mapsto Ux$, where $U$ is an $n \times n$ matrix.

$U$ is called the **transition matrix** from $v_1, v_2, \ldots, v_n$ to $u_1, u_2, \ldots, u_n$. Columns of $U$ are coordinates of the vectors $v_1, v_2, \ldots, v_n$ with respect to the basis $u_1, u_2, \ldots, u_n$. 
Change of coordinates for a linear operator

Let \( L : V \to V \) be a linear operator on a vector space \( V \).

Let \( A \) be the matrix of \( L \) relative to a basis \( a_1, a_2, \ldots, a_n \) for \( V \). Let \( B \) be the matrix of \( L \) relative to another basis \( b_1, b_2, \ldots, b_n \) for \( V \).

Let \( U \) be the transition matrix from the basis \( a_1, a_2, \ldots, a_n \) to \( b_1, b_2, \ldots, b_n \).

\[
\begin{array}{ccc}
\text{a-coordinates of } v & \xrightarrow{A} & \text{a-coordinates of } L(v) \\
U & \downarrow & U \\
\text{b-coordinates of } v & \xrightarrow{B} & \text{b-coordinates of } L(v)
\end{array}
\]

It follows that \( UAx = BUx \) for all \( x \in \mathbb{F}^n \) \( \implies \) \( UA = BU \).

Then \( A = U^{-1}BU \) and \( B = UAU^{-1} \).
Problem. Consider a linear operator \( L : \mathbb{F}^2 \rightarrow \mathbb{F}^2 \),

\[
L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

Find the matrix of \( L \) with respect to the basis
\( \mathbf{v}_1 = (3, 1), \mathbf{v}_2 = (2, 1) \).

Let \( S \) be the matrix of \( L \) with respect to the standard basis, \( N \) be the matrix of \( L \) with respect to the basis \( \mathbf{v}_1, \mathbf{v}_2 \), and \( U \) be the transition matrix from \( \mathbf{v}_1, \mathbf{v}_2 \) to \( \mathbf{e}_1, \mathbf{e}_2 \). Then \( N = U^{-1}SU \).

\[
S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},
\]

\[
N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.
\]
**Similarity**

*Definition.* An $n \times n$ matrix $B$ is said to be **similar** to an $n \times n$ matrix $A$ if $B = S^{-1}AS$ for some nonsingular $n \times n$ matrix $S$.

*Remark.* Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{F}^n$ with respect to different bases.

*Theorem*  Similarity is an *equivalence relation*, which means that

(i) any square matrix $A$ is similar to itself;
(ii) if $B$ is similar to $A$, then $A$ is similar to $B$;
(iii) if $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$. 
**Theorem**  Let $V, W$ be finite-dimensional vector spaces and $f : V \to W$ be a linear map. Then one can choose bases for $V$ and $W$ so that the respective matrix of $f$ is has the block form

$$
\begin{pmatrix}
I_r & 0 \\
0 & O
\end{pmatrix},
$$

where $r$ is the rank of $f$.

**Example.** With a suitable choice of bases, any linear map $f : \mathbb{F}^3 \to \mathbb{F}^2$ has one of the following matrices:

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
$$
Proof of the theorem:
Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for the null-space $\mathcal{N}(f)$.
Extend it to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_r$ for $V$.
Then $f(\mathbf{u}_1), f(\mathbf{u}_2), \ldots, f(\mathbf{u}_r)$ is a basis for the range $\mathcal{R}(f)$.
Extend it to a basis $f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l$ for $W$.

Now the matrix of $f$ with respect to bases $[\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_k]$ and $[f(\mathbf{u}_1), \ldots, f(\mathbf{u}_r), \mathbf{w}_1, \ldots, \mathbf{w}_l]$ is
\[
\begin{pmatrix}
I_r & \mathbf{O} \\
\mathbf{O} & \mathbf{O}
\end{pmatrix}.
\]
Definition. A map \( f : V_1 \to V_2 \) is one-to-one if it maps different elements from \( V_1 \) to different elements in \( V_2 \). The map \( f \) is onto if any element \( y \in V_2 \) is represented as \( f(x) \) for some \( x \in V_1 \).

If the map \( f \) is both one-to-one and onto, then the inverse map \( f^{-1} : V_2 \to V_1 \) is well defined.

Now let \( V_1, V_2 \) be vector spaces and \( L : V_1 \to V_2 \) be a linear map.

**Theorem (i)** The linear map \( L \) is one-to-one if and only if \( \mathcal{N}(L) = \{0\} \).

**(ii)** The linear map \( L \) is onto if \( \mathcal{R}(L) = V_2 \).

**(iii)** If the linear map \( L \) is both one-to-one and onto, then the inverse map \( L^{-1} \) is also linear.
Definition. A linear map $L : V_1 \rightarrow V_2$ is called an isomorphism of vector spaces if it is both one-to-one and onto.

The vector space $V_1$ is said to be isomorphic to $V_2$ if there exists an isomorphism $L : V_1 \rightarrow V_2$.

The word “isomorphism” applies when two complex structures can be mapped onto each other, in such a way that to each part of one structure there is a corresponding part in the other structure, where “corresponding” means that the two parts play similar roles in their respective structures.
Alternative notation

**General maps**

one-to-one .......................................................... injective
onto .......................................................... surjective
one-to-one and onto ......................................... bijective

**Linear maps**

any map .......................................................... homomorphism
one-to-one .......................................................... monomorphism
onto .......................................................... epimorphism
one-to-one and onto ......................................... isomorphism

**Linear self-maps**

any map .......................................................... endomorphism
one-to-one and onto ......................................... automorphism
Examples of isomorphism

- $\mathcal{M}_{1,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,1}(\mathbb{F})$.
  Isomorphism: $(x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

- $\mathcal{M}_{2,2}(\mathbb{F})$ is isomorphic to $\mathbb{F}^4$.
  Isomorphism: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$.

- $\mathcal{M}_{2,3}(\mathbb{F})$ is isomorphic to $\mathcal{M}_{3,2}(\mathbb{F})$.
  Isomorphism: $A \mapsto A^t$.

- The plane $z = 0$ in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.
  Isomorphism: $(x, y, 0) \mapsto (x, y)$. 
Examples of isomorphism

- $\mathcal{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.
  Isomorphism: $a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n)$.

- $\mathcal{P}$ is isomorphic to $\mathbb{R}_0^\infty$.
  Isomorphism:
  
  $a_0 + a_1 x + \cdots + a_n x^n \mapsto (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$.

- $\mathcal{M}_{m,n}(\mathbb{F})$ is isomorphic to $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.
  Isomorphism: $A \mapsto L_A$, where $L_A(x) = Ax$.

- Any vector space $V$ of dimension $n$ is isomorphic to $\mathbb{F}^n$.
  Isomorphism: $v \mapsto [v]_\alpha$, where $\alpha$ is a basis for $V$. 
Isomorphism and dimension

*Definition.* Two sets $S_1$ and $S_2$ are said to be of the same **cardinality** if there exists a bijective map $f : S_1 \to S_2$.

**Theorem 1** All bases of a fixed vector space $V$ are of the same cardinality.

**Theorem 2** Two vector spaces are isomorphic if and only if their bases are of the same cardinality. In particular, a vector space $V$ is isomorphic to $\mathbb{F}^n$ if and only if $\dim V = n$.

*Remark.* For a finite set, the cardinality is a synonym for the number of its elements. For an infinite set, the cardinality is a more sophisticated notion. For example, $\mathbb{R}^\infty$ and $\mathcal{P}$ are both infinite-dimensional vector spaces but they are not isomorphic.