Lecture 14:
General linear equations.
Elementary matrices.
General linear equations

Definition. A **linear equation** is an equation of the form

\[ L(x) = b, \]

where \( L : V \rightarrow W \) is a linear mapping, \( b \) is a given vector from \( W \), and \( x \) is an unknown vector from \( V \).

The range of \( L \) is the set of all vectors \( b \in W \) such that the equation \( L(x) = b \) has a solution.

The null-space of \( L \) is the solution set of the **homogeneous** linear equation \( L(x) = 0 \).

**Theorem** If the linear equation \( L(x) = b \) is solvable and \( \dim \mathcal{N}(L) < \infty \), then the general solution is

\[ x_0 + t_1 v_1 + \cdots + t_k v_k, \]

where \( x_0 \) is a particular solution, \( v_1, \ldots, v_k \) is a basis for the null-space \( \mathcal{N}(L) \), and \( t_1, \ldots, t_k \) are arbitrary scalars.
Example. \[ \begin{cases} x + y + z = 4, \\ x + 2y = 3. \end{cases} \]

\[ L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]

Linear equation: \[ L(x) = b, \] where \[ b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}. \]

\[ \begin{cases} x + y + z = 4 \\ x + 2y = 3 \end{cases} \iff \begin{cases} x + y + z = 4 \\ y - z = -1 \end{cases} \iff \begin{cases} x + 2z = 5 \\ y - z = -1 \end{cases} \iff \begin{cases} x = 5 - 2z \\ y = -1 + z \end{cases} \]

\[ (x, y, z) = (5 - 2t, -1 + t, t) = (5, -1, 0) + t(-2, 1, 1). \]
**Example.** \( u'''(x) - 2u''(x) + u'(x) = e^{2x} \).

Linear operator \( L : C^3(\mathbb{R}) \to C(\mathbb{R}), \ Lu = u''' - 2u'' + u' \).

Linear equation: \( Lu = b \), where \( b(x) = e^{2x} \).

According to the theory of differential equations, the initial value problem

\[
\begin{align*}
  u'''(x) - 2u''(x) + u'(x) &= g(x), \quad x \in \mathbb{R}, \\
  u(a) &= b_0, \\
  u'(a) &= b_1, \\
  u''(a) &= b_2
\end{align*}
\]

has a unique solution for any \( g \in C(\mathbb{R}) \) and any \( b_0, b_1, b_2 \in \mathbb{R} \). It follows that \( L(C^3(\mathbb{R})) = C(\mathbb{R}) \).

Also, the initial data evaluation \( I(u) = (u(a), u'(a), u''(a)) \), which is a linear mapping \( I : C^3(\mathbb{R}) \to \mathbb{R}^3 \), is one-to-one and onto when restricted to \( \mathcal{N}(L) \). Hence \( \dim \mathcal{N}(L) = 3 \).

It is easy to check that \( L(xe^x) = L(e^x) = L(1) = 0 \). One can also show that \( xe^x, e^x, \) and \( 1 \) are linearly independent.
Example. \( u'''(x) - 2u''(x) + u'(x) = e^{2x} \).

Linear operator \( L : C^3(\mathbb{R}) \to C(\mathbb{R}), \)
\( Lu = u''' - 2u'' + u'. \)

Linear equation: \( Lu = b, \) where \( b(x) = e^{2x}. \)

It follows from the previous slide that functions \( xe^x, e^x \) and 1 form a basis for the null-space of \( L. \) It remains to find a particular solution.

\( L(e^{2x}) = 8e^{2x} - 2(4e^{2x}) + 2e^{2x} = 2e^{2x}. \)

Since \( L \) is a linear operator, \( L\left(\frac{1}{2}e^{2x}\right) = e^{2x}. \)

Particular solution: \( u_0(x) = \frac{1}{2}e^{2x}. \)

Thus the general solution is
\[ u(x) = \frac{1}{2}e^{2x} + t_1xe^x + t_2e^x + t_3. \]
Elementary row operations for matrices:
(1) to interchange two rows;
(2) to multiply a row by a nonzero scalar;
(3) to add the $i$th row multiplied by some scalar $r$ to the $j$th row.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Similarly, we define three types of elementary column operations.
Elementary row operations

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
= \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m
\end{pmatrix},
\]

where \( v_i = (a_{i1} a_{i2} \ldots a_{in}) \) is a row vector.
Elementary row operations

Operation 1: to interchange the $i$th row with the $j$th row:

$$\begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_j \\ \vdots \\ v_m \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ \vdots \\ v_j \\ \vdots \\ v_i \\ \vdots \\ v_m \end{pmatrix}$$
Elementary row operations

**Operation 2:** to multiply the $i$th row by $r \neq 0$:

\[
\begin{pmatrix}
  \mathbf{v}_1 \\
  \vdots \\
  \mathbf{v}_i \\
  \vdots \\
  \mathbf{v}_m
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \mathbf{v}_1 \\
  \vdots \\
  r\mathbf{v}_i \\
  \vdots \\
  \mathbf{v}_m
\end{pmatrix}
\]
Elementary row operations

*Operation 3:* to add the $i$th row multiplied by $r$ to the $j$th row:

\[
\begin{pmatrix}
  v_1 \\
  \vdots \\
  v_i \\
  \vdots \\
  v_j \\
  \vdots \\
  v_m
\end{pmatrix} \rightarrow \begin{pmatrix}
  v_1 \\
  \vdots \\
  v_i \\
  \vdots \\
  v_j + r v_i \\
  \vdots \\
  v_m
\end{pmatrix}
\]
**Theorem**  Any elementary row operation can be simulated as left multiplication by a certain matrix.

**Examples.**

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
= 
\begin{pmatrix}
a_1 & a_2 & a_3 \\
2b_1 & 2b_2 & 2b_3 \\
c_1 & c_2 & c_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
= 
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1+3a_1 & b_2+3a_2 & b_3+3a_3 \\
c_1 & c_2 & c_3
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
= 
\begin{pmatrix}
a_1 & a_2 & a_3 \\
c_1 & c_2 & c_3 \\
b_1 & b_2 & b_3
\end{pmatrix}.
\]
Elementary matrices

To obtain the matrix $EA$ from $A$, interchange the $i$th row with the $j$th row. To obtain $AE$ from $A$, interchange the $i$th column with the $j$th column.
Elementary matrices

\[ E = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \text{ row } \#i \]

To obtain the matrix \( EA \) from \( A \), multiply the \( i \)th row by \( r \). To obtain the matrix \( AE \) from \( A \), multiply the \( i \)th column by \( r \).
Elementary matrices

\[ E = \begin{pmatrix}
1 & \cdots & O \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
0 & \cdots & r & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \ddots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 1
\end{pmatrix} \]

row \#i

row \#j

To obtain the matrix \( EA \) from \( A \), add \( r \) times the \( i \)th row to the \( j \)th row. To obtain the matrix \( AE \) from \( A \), add \( r \) times the \( j \)th column to the \( i \)th column.
Notice that the elementary matrix $E_\sigma$ simulating an elementary row operation $\sigma$ is obtained by applying $\sigma$ to the identity matrix. In particular, this implies that $E_\sigma$ is unique.

**Theorem** Any elementary row operation $\sigma_1$ can be undone by applying another elementary row operation $\sigma_2$. Moreover, the operation $\sigma_1$ will undo the operation $\sigma_2$.

**Corollary** Elementary matrices are invertible.

*Proof:* Let $E$ be an elementary matrix simulating an elementary row operation $\sigma$. Let $\tau$ be the operation such that $\sigma$ and $\tau$ undo each other. The operation $\tau$ is simulated as left multiplication by some matrix $E_0$. Then $E_0EA = EE_0A = A$ for any matrix $A$. When $A = I$, we get $E_0E = EE_0 = I$. 