MATH 423
Linear Algebra II

Lecture 18:
Determinants (continued).
Determinants

**Determinant** is a scalar assigned to each square matrix.

*Notation.* The determinant of a matrix 
\[ A = (a_{ij})_{1 \leq i, j \leq n} \] 
is denoted \( \det A \) or

\[
\begin{vmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & a_{nn}
\end{vmatrix}.
\]

**Principal property:** \( \det A \neq 0 \) if and only if a system of linear equations with the coefficient matrix \( A \) has a unique solution. Equivalently, \( \det A \neq 0 \) if and only if the matrix \( A \) is invertible.
**Definition in low dimensions**

**Definition.** \( \det (a) = a, \) \[
\begin{vmatrix}
a & b \\
c & d
\end{vmatrix} = ad - bc,
\]
\[
\begin{vmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}
- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
\]

**+:** \[
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix},
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix},
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix}.
\]

**−:** \[
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix},
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix},
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & *
\end{pmatrix}.
\]
Examples: $3 \times 3$ matrices

\[
\begin{vmatrix}
3 & -2 & 0 \\
1 & 0 & 1 \\
-2 & 3 & 0
\end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3
\]

\[
-0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,
\]

\[
\begin{vmatrix}
1 & 4 & 6 \\
0 & 2 & 5 \\
0 & 0 & 3
\end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0
\]

\[
-6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.
\]
Let us try to find a solution of a general system of 2 linear equations in 2 variables:

\[
\begin{align*}
\begin{cases}
a_{11}x + a_{12}y &= b_1, \\
a_{21}x + a_{22}y &= b_2.
\end{cases}
\end{align*}
\]

Solve the 1st equation for \(x\): \(x = (b_1 - a_{12}y)/a_{11}\).
Substitute into the 2nd equation:

\[a_{21}(b_1 - a_{12}y)/a_{11} + a_{22}y = b_2.\]

Solve for \(y\):

\[y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.\]

Back substitution: \(x = (b_1 - a_{12}y)/a_{11} = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.\)

Thus

\[
\begin{align*}
x &= \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, & \quad y &= \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.
\end{align*}
\]
General definition

The general definition of the determinant is quite complicated as there is no simple explicit formula.

There are several approaches to defining determinants.

**Approach 1 (original):** an explicit (but very complicated) formula.

**Approach 2 (axiomatic):** we formulate properties that the determinant should have.

**Approach 3 (inductive):** the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n - 1) \times (n - 1)$ matrices.
Original definition of determinant

**Definition.** If $A = (a_{ij})$ is an $n \times n$ matrix then

$$\det A = \sum_{\pi \in S_n} \text{sgn}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

where $\pi$ runs over $S_n$, the set of all permutations of \{1, 2, \ldots, n\}, and $\text{sgn}(\pi)$ denotes the sign of the permutation $\pi$.

**Remarks.** • A **permutation** of the set \{1, 2, \ldots, n\} is an invertible mapping of this set onto itself. There are $n!$ such mappings.

• The **sign** $\text{sgn}(\pi)$ can be 1 or $-1$. Its definition is rather complicated.
\( M_{n,n}(F) \): the set of \( n \times n \) matrices with entries in \( F \).

**Theorem**  There exists a unique function
\( \det : M_{n,n}(F) \to F \) (called the determinant) with
the following properties:

**(D1)** if we interchange two rows of a matrix, the
determinant changes its sign;

**(D2)** if a row of a matrix is multiplied by a scalar
\( r \), the determinant is also multiplied by \( r \);

**(D3)** if we add a row of a matrix multiplied by a
scalar to another row, the determinant remains the
same;

**(D4)** \( \det I = 1 \).
Corollary 1  Suppose \( A \) is a square matrix and \( B \) is obtained from \( A \) applying elementary row operations. Then \( \det A = 0 \) if and only if \( \det B = 0 \).

Corollary 2  \( \det B = 0 \) whenever the matrix \( B \) has a zero row.

*Hint:* Multiply the zero row by the zero scalar.

Corollary 3  \( \det A = 0 \) if and only if the matrix \( A \) is not invertible.

*Idea of the proof:* Let \( B \) be the reduced row echelon form of \( A \). If \( A \) is invertible then \( B = I \); otherwise \( B \) has a zero row.

*Remark.* The same argument proves that properties (D1)–(D4) are enough to compute any determinant.
Row echelon form of a square matrix $A$:

$$
\begin{pmatrix}
\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \ast & \\
\ast & \ast & \\
\ast & \\
\ast
\end{array}
\end{pmatrix}
$$

$$
\begin{pmatrix}
\begin{array}{cccccc}
\ast & \ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \ast & \\
\ast & \ast & \ast & \\
\ast & \\
\ast & \\
\ast
\end{array}
\end{pmatrix}
$$

$\det A \neq 0$ \hspace{1cm} $\det A = 0$
Other properties of determinants

• If a matrix \( A \) has two identical rows then \( \det A = 0 \).

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  a_1 & a_2 & a_3 \\
\end{vmatrix}
= \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  0 & 0 & 0 \\
\end{vmatrix}
= 0
\]

• If a matrix \( A \) has two proportional rows then \( \det A = 0 \).

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  ra_1 & ra_2 & ra_3 \\
\end{vmatrix}
= r
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  a_1 & a_2 & a_3 \\
\end{vmatrix}
= 0
\]

• If the rows of \( A \) are linearly dependent then \( \det A = 0 \) (as in this case \( A \) is not invertible).
Definition. A square matrix $A = (a_{ij})$ is called **diagonal** if all entries off the main diagonal are zeros: $a_{ij} = 0$ whenever $i \neq j$. The matrix $A$ is called **upper triangular** if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever $i > j$.

- If $A$ is a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$ then $\det A = d_1 d_2 \ldots d_n$.

- The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  0 & a_{22} & a_{23} \\
  0 & 0 & a_{33}
\end{vmatrix} = a_{11} a_{22} a_{33}
\]
Additive law for rows

- Suppose that matrices $X, Y, Z$ are identical except for the $i$th row and the $i$th row of $Z$ is the sum of the $i$th rows of $X$ and $Y$.

  Then $\det Z = \det X + \det Y$.

\[
\begin{vmatrix}
  a_1 + a'_1 & a_2 + a'_2 & a_3 + a'_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
= \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
+ \begin{vmatrix}
  a'_1 & a'_2 & a'_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

Together with property (D2), this means that the determinant depends linearly on each row of a matrix.
Submatrices

*Definition.* Given a matrix $A$, a $k \times k$ submatrix of $A$ is a matrix obtained by specifying $k$ columns and $k$ rows of $A$ and deleting the other columns and rows.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
10 & 20 & 30 & 40 \\
3 & 5 & 7 & 9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
* & 2 & * & 4 \\
* & * & * & * \\
* & 5 & * & 9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & 4 \\
5 & 9
\end{pmatrix}
\]

*Theorem* Suppose $A$ is a matrix of rank $m$. Then $A$ admits a $k \times k$ submatrix with nonzero determinant if and only if $0 < k \leq m$. 
Row and column expansions

Given an \( n \times n \) matrix \( A = (a_{ij}) \), let \( M_{ij} \) denote the \((n - 1) \times (n - 1)\) submatrix obtained by deleting the \( i \)th row and the \( j \)th column of \( A \).

**Theorem**  For any \( 1 \leq k, m \leq n \) we have that

\[
\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},
\]

\((\text{expansion by } k\text{th row})\)

\[
\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.
\]

\((\text{expansion by } m\text{th column})\)
Signs for row/column expansions

\[
\begin{pmatrix}
+ & - & + & - & \cdots \\
- & + & - & + & \cdots \\
+ & - & + & - & \cdots \\
- & + & - & + & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Example. \[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}. \]

Expansion by the 1st row:

\[
\begin{pmatrix} 1 & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & 2 & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & 3 \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}
\]

\[
\det A = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}
\]

\[
= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0.
\]
Example. \( A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \).

Expansion by the 2nd column:

\[
\begin{pmatrix} * & 2 & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \begin{pmatrix} 1 & * & 3 \\ * & 5 & * \\ 7 & * & 9 \end{pmatrix} \begin{pmatrix} 1 & * & 3 \\ * & 8 & * \\ 4 & 9 & 6 \end{pmatrix}
\]

\[
\det A = -2 \left| \begin{array}{cc} 4 & 6 \\ 7 & 9 \end{array} \right| + 5 \left| \begin{array}{cc} 1 & 3 \\ 7 & 9 \end{array} \right| - 8 \left| \begin{array}{cc} 1 & 3 \\ 4 & 6 \end{array} \right|
\]

\[
= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0.
\]
Example. \[ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \] 

Subtract the 1st row from the 2nd row and from the 3rd row:

\[
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0
\]

since the last matrix has two proportional rows.