MATH 423
Linear Algebra II

Lecture 20:
Geometry of linear transformations.
Eigenvalues and eigenvectors.
Characteristic polynomial.
Geometric properties of determinants

- **2×2 determinants and plane geometry**
  Let $P$ be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $v_1, v_2 \in \mathbb{R}^2$ are represented by adjacent sides of $P$. Then $\text{area}(P) = |\det A|$, where $A = (v_1, v_2)$, a matrix whose columns are $v_1$ and $v_2$.

  Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(v) = Av$ for any column vector $v$. Then $\text{area}(L_A(D)) = |\det A| \text{area}(D)$ for any bounded domain $D$.

- **3×3 determinants and space geometry**
  Let $\Pi$ be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ are represented by adjacent edges of $\Pi$. Then $\text{volume}(\Pi) = |\det B|$, where $B = (v_1, v_2, v_3)$, a matrix whose columns are $v_1, v_2, v_3$.

  Similarly, $\text{volume}(L_B(D)) = |\det B| \text{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$. 
volume(\(\Pi\)) = |\det B|, where \(B = (v_1, v_2, v_3)\). Note that the parallelepiped \(\Pi\) is the image under \(L_B\) of a unit cube whose adjacent edges are \(e_1, e_2, e_3\).

The triple \(v_1, v_2, v_3\) obeys the right-hand rule. We say that \(L_B\) preserves orientation if it preserves the hand rule for any basis. This is the case if and only if \(\det B > 0\).
Any linear operator $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a $2 \times 2$ matrix: $L(x) = Ax$ or

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Linear transformations corresponding to particular matrices can have various geometric properties.
Texture Rotation by $90^\circ$

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}

Rotation by 45°
Reflection about the vertical axis

\[ A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]
Reflection about the line $x - y = 0$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$

Horizontal shear
\[ A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \]
\[ A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix} \]
Texture

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

Vertical projection on the horizontal axis
Texture

\[ A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \]

Horizontal projection on the line \( x + y = 0 \)
\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
Definition. Let $A \in M_{n,n}(\mathbb{F})$. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of the matrix $A$ if $Av = \lambda v$ for a nonzero column vector $v \in \mathbb{F}^n$.

The vector $v$ is called an **eigenvector** of $A$ belonging to (or associated with) the eigenvalue $\lambda$.

Remarks. • Alternative notation: eigenvalue = *characteristic value*,
eigenvector = *characteristic vector*.

• The zero vector is never considered an eigenvector.
Example. \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

Hence \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is an eigenvector of \( A \) belonging to the eigenvalue \( 1 \), while \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) is an eigenvector of \( A \) belonging to the eigenvalue \( -1 \).
Eigenspaces

Let $A$ be an $n \times n$ matrix. Let $\mathbf{v}$ be an eigenvector of $A$ belonging to an eigenvalue $\lambda$. Then

$$A\mathbf{v} = \lambda \mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}.$$ 

Hence $\mathbf{v} \in \mathcal{N}(A - \lambda I)$, the null-space of the matrix $A - \lambda I$.

Conversely, if $\mathbf{x} \in \mathcal{N}(A - \lambda I)$ then $A\mathbf{x} = \lambda \mathbf{x}$. Thus the eigenvectors of $A$ belonging to the eigenvalue $\lambda$ are nonzero vectors from $\mathcal{N}(A - \lambda I)$.

**Definition.** If $\mathcal{N}(A - \lambda I) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of the matrix $A$ corresponding to the eigenvalue $\lambda$. 
How to find eigenvalues and eigenvectors?

**Theorem**  Given a square matrix $A$ and a scalar $\lambda$, the following conditions are equivalent:

- $\lambda$ is an eigenvalue of $A$,
- $\mathcal{N}(A - \lambda I) \neq \{0\}$,
- the matrix $A - \lambda I$ is not invertible,
- $\det(A - \lambda I) = 0$.

**Definition.** $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix $A$.

Eigenvalues $\lambda$ of $A$ are roots of the characteristic equation. Associated eigenvectors of $A$ are nonzero solutions of the equation $(A - \lambda I)x = 0$. 
Example. \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

\[
\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\
= (a - \lambda)(d - \lambda) - bc \\
= \lambda^2 - (a + d)\lambda + (ad - bc).
\]
Example. \( A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \).

\[
det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}
\]

\[= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,\]

where \( c_1 = a_{11} + a_{22} + a_{33} \) (the trace of \( A \)),

\[
c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},
\]

\( c_3 = \det A. \)
**Theorem.** Let $A = (a_{ij})$ be an $n \times n$ matrix. Then $\det(A - \lambda I)$ is a polynomial of $\lambda$ of degree $n$: 
$\det(A - \lambda I) = (-1)^n \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1} \lambda + c_n$.

Furthermore, $(-1)^{n-1} c_1 = a_{11} + a_{22} + \cdots + a_{nn}$ and $c_n = \det A$.

**Definition.** The polynomial $p(\lambda) = \det(A - \lambda I)$ is called the **characteristic polynomial** of the matrix $A$.

**Corollary** Any $n \times n$ matrix has at most $n$ eigenvalues.
Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Characteristic equation: $\begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$.

$(2 - \lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \lambda_2 = 3$.

$(A - I)x = 0 \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0$.

The general solution is $(-t, t) = t(-1, 1), \ t \in \mathbb{R}$. Thus $v_1 = (-1, 1)$ is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by $v_1$. 
\[(A - 3I)x = 0 \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

\[\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.\]

The general solution is \((t, t) = t(1, 1), \quad t \in \mathbb{R}.\)

Thus \(v_2 = (1, 1)\) is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by \(v_2.\)
Summary. \[
A = \begin{pmatrix}
2 & 1 \\
1 & 2
\end{pmatrix}.
\]

- The matrix \( A \) has two eigenvalues: 1 and 3.
- The eigenspace of \( A \) associated with the eigenvalue 1 is the line \( t(−1, 1) \).
- The eigenspace of \( A \) associated with the eigenvalue 3 is the line \( t(1, 1) \).
- Eigenvectors \( \mathbf{v}_1 = (−1, 1) \) and \( \mathbf{v}_2 = (1, 1) \) of the matrix \( A \) are orthogonal and form a basis for \( \mathbb{R}^2 \).
- Geometrically, the mapping \( \mathbf{x} \mapsto A\mathbf{x} \) is a stretch by a factor of 3 away from the line \( x + y = 0 \) in the orthogonal direction.
**Eigenvalues and eigenvectors of an operator**

*Definition.* Let $V$ be a vector space and $L : V \to V$ be a linear operator. A scalar $\lambda$ is called an **eigenvalue** of the operator $L$ if $L(v) = \lambda v$ for a nonzero vector $v \in V$. The vector $v$ is called an **eigenvector** of $L$ associated with the eigenvalue $\lambda$.

If $V = \mathbb{F}^n$ then the linear operator $L$ is given by $L(x) = Ax$, where $A$ is an $n \times n$ matrix. In this case, eigenvalues and eigenvectors of the operator $L$ are precisely eigenvalues and eigenvectors of the matrix $A$.

For a general finite-dimensional vector space $V$, we choose an ordered basis $\alpha$. Then

$$L(v) = \lambda v \iff [L]_\alpha[v]_\alpha = \lambda[v]_\alpha.$$ 

Hence the eigenvalues of $L$ coincide with those of the matrix $[L]_\alpha$. Moreover, the associated eigenvectors of $[L]_\alpha$ are coordinates of the eigenvectors of $L$. 