Lecture 30:
The Gram-Schmidt process.
Orthogonal complement.
Orthogonal sets

Let \( V \) be an inner product space with an inner product \( \langle \cdot, \cdot \rangle \) and the induced norm \( \|v\| = \sqrt{\langle v, v \rangle} \).

**Definition.** Vectors \( v_1, v_2, \ldots, v_k \in V \) form an **orthogonal set** if they are orthogonal to each other: \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \).

If, in addition, all vectors are of unit norm, \( \|v_i\| = 1 \), then \( v_1, v_2, \ldots, v_k \) is called an **orthonormal set**.

**Theorem** Any orthogonal set of nonzero vectors is linearly independent.
Orthogonal basis

**Theorem**  If \( v_1, v_2, \ldots, v_n \) is an orthogonal basis for an inner product space \( V \), then

\[
x = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle x, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots + \frac{\langle x, v_n \rangle}{\langle v_n, v_n \rangle} v_n
\]

for any vector \( x \in V \).

**Corollary**  If \( v_1, v_2, \ldots, v_n \) is an orthonormal basis for an inner product space \( V \), then

\[
x = \langle x, v_1 \rangle v_1 + \langle x, v_2 \rangle v_2 + \cdots + \langle x, v_n \rangle v_n
\]

for any vector \( x \in V \).
**Orthogonal projection**

**Theorem** Let $V$ be an inner product space and $V_0$ be a finite-dimensional subspace of $V$. Then any vector $x \in V$ is uniquely represented as $x = p + o$, where $p \in V_0$ and $o \perp V_0$.

The component $p$ is called the **orthogonal projection** of the vector $x$ onto the subspace $V_0$. 
Theorem  Let $V$ be an inner product space and $V_0$ be a finite-dimensional subspace of $V$. Then any vector $x \in V$ is uniquely represented as $x = p + o$, where $p \in V_0$ and $o \perp V_0$.

Proof of uniqueness: Suppose $x = p + o = p' + o'$, where $p, p' \in V_0$, $o \perp V_0$, $o' \perp V_0$. Then $o - o' = p' - p \in V_0$. It follows that $\langle o, o - o' \rangle = \langle o', o - o' \rangle = 0$. Hence $\langle o - o', o - o' \rangle = \langle o, o - o' \rangle - \langle o', o - o' \rangle = 0$ so that $o - o' = 0$. Thus $o = o'$, then $p = p'$.

Proof of existence in the case $V_0$ admits an orthogonal basis: Suppose $v_1, v_2, \ldots, v_n$ is an orthogonal basis for $V_0$. Let

$$p = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle x, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots + \frac{\langle x, v_n \rangle}{\langle v_n, v_n \rangle} v_n$$

and $o = x - p$. By construction, $x = p + o$ and $p \in V_0$. Just as in the previous lecture, we obtain that $\langle p, v_i \rangle = \langle x, v_i \rangle$ for $1 \leq i \leq n$. Then $o \perp v_i$ for all $i$, which implies that $o \perp V_0$. 
The Gram-Schmidt orthogonalization process

Let \( V \) be a vector space with an inner product. Suppose \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) is a basis for \( V \). Let

\[
\mathbf{v}_1 = \mathbf{x}_1,
\]

\[
\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,
\]

\[
\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,
\]

\[
\ldots
\]

\[
\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \cdots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.
\]

Then \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) is an orthogonal basis for \( V \).
\[ \text{Span}(\mathbf{v}_1, \mathbf{v}_2) = \text{Span}(\mathbf{x}_1, \mathbf{x}_2) \]

\[ \mathbf{p}_3 = \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \]
Any basis \( x_1, x_2, \ldots, x_n \) \quad \rightarrow \quad Orthogonal basis \( v_1, v_2, \ldots, v_n \)

Properties of the Gram-Schmidt process:

- \( v_k = x_k - (\alpha_1 x_1 + \cdots + \alpha_{k-1} x_{k-1}) \), 1 \leq k \leq n;
- the span of \( v_1, \ldots, v_k \) is the same as the span of \( x_1, \ldots, x_k \);
- \( v_k \) is orthogonal to \( x_1, \ldots, x_{k-1} \);
- \( v_k = x_k - p_k \), where \( p_k \) is the orthogonal projection of the vector \( x_k \) on the subspace spanned by \( x_1, \ldots, x_{k-1} \).
Normalization

Let $V$ be a vector space with an inner product. Suppose $v_1, v_2, \ldots, v_n$ is an orthogonal basis for $V$. Let $w_1 = \frac{v_1}{\|v_1\|}, w_2 = \frac{v_2}{\|v_2\|}, \ldots, w_n = \frac{v_n}{\|v_n\|}$. Then $w_1, w_2, \ldots, w_n$ is an orthonormal basis for $V$.

**Theorem**  Any finite-dimensional vector space with an inner product has an orthonormal basis.

**Remark.** An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.
Orthogonal complement

**Definition.** Let $S$ be a nonempty subset of an inner product space $W$. The **orthogonal complement** of $S$, denoted $S^\perp$, is the set of all vectors $x \in W$ that are orthogonal to $S$.

**Theorem**  Let $V$ be a subspace of $W$. Then

(i) $V^\perp$ is a closed subspace of $W$;
(ii) $V \subset (V^\perp)^\perp$;
(iii) $V \cap V^\perp = \{0\}$;
(iv) $\dim V + \dim V^\perp = \dim W$ if $\dim W < \infty$;
(v) if $\dim V < \infty$, then $V \oplus V^\perp = W$, that is, any vector $x \in W$ is (uniquely) represented as $x = p + o$, where $p \in V$ and $o \in V^\perp$.

**Remark.** The orthogonal projection onto a subspace $V$ is well defined if and only if $V \oplus V^\perp = W$. 
Suppose $V$ is a subspace of an inner product space $W$ such that $V \oplus V^\perp = W$. Let $p$ be the orthogonal projection of a vector $x \in W$ onto $V$.

**Theorem**  $\|x - v\| > \|x - p\|$ for any $v \neq p$ in $V$.

**Proof:** Let $o = x - p$, $o_1 = x - v$, and $v_1 = p - v$. Then $o_1 = o + v_1$, $v_1 \in V$, and $v_1 \neq 0$. Since $o \perp V$, it follows that $\langle o, v_1 \rangle = 0$.

$$
\|o_1\|^2 = \langle o_1, o_1 \rangle = \langle o + v_1, o + v_1 \rangle \\
= \langle o, o \rangle + \langle v_1, o \rangle + \langle o, v_1 \rangle + \langle v_1, v_1 \rangle \\
= \langle o, o \rangle + \langle v_1, v_1 \rangle = \|o\|^2 + \|v_1\|^2 > \|o\|^2.
$$

Thus $\|x - p\| = \min_{v \in V} \|x - v\|$ is the **distance** from the vector $x$ to the subspace $V$. 
Problem. Find the distance from the point \( y = (0, 0, 0, 1) \) to the subspace \( V \subset \mathbb{R}^4 \) spanned by vectors \( x_1 = (1, -1, 1, -1), \ x_2 = (1, 1, 3, -1), \) and \( x_3 = (-3, 7, 1, 3). \)

Let us apply the Gram-Schmidt process to vectors \( x_1, x_2, x_3, y \). We should obtain an orthogonal set \( v_1, v_2, v_3, v_4 \). The desired distance will be \( |v_4| \).
$\mathbf{x}_1 = (1, -1, 1, -1), \quad \mathbf{x}_2 = (1, 1, 3, -1), \quad \mathbf{x}_3 = (-3, 7, 1, 3), \quad \mathbf{y} = (0, 0, 0, 1).$

$\mathbf{v}_1 = \mathbf{x}_1 = (1, -1, 1, -1),$

$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 = (1, 1, 3, -1) - \frac{4}{4}(1, -1, 1, -1)$

$= (0, 2, 2, 0),$

$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\mathbf{x}_3}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2$

$= (-3, 7, 1, 3) - \frac{-12}{4}(1, -1, 1, -1) - \frac{16}{8}(0, 2, 2, 0)$

$= (0, 0, 0, 0).$
The Gram-Schmidt process can be used to check linear independence of vectors!

The vector \( \mathbf{x}_3 \) is a linear combination of \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \).

\( \mathcal{V} \) is a plane, not a 3-dimensional subspace.

We should orthogonalize vectors \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \).

\[
\tilde{\mathbf{v}}_3 = \mathbf{y} - \frac{\langle \mathbf{y}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{y}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2
\]

\[
= (0, 0, 0, 1) - \frac{-1}{4} (1, -1, 1, -1) - \frac{0}{8} (0, 2, 2, 0)
= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right).
\]

\[
|\tilde{\mathbf{v}}_3| = \left| \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right) \right| = \frac{1}{4} \left| (1, -1, 1, 3) \right| = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}.
\]