MATH 423
Linear Algebra II

Lecture 31:
Dual space.
Adjoint operator.
Dual space

Let $V$ be a vector space over a field $\mathbb{F}$.

**Definition.** The vector space $\mathcal{L}(V, \mathbb{F})$ of all linear functionals $\ell : V \to \mathbb{F}$ is called the **dual space** of $V$ (denoted $V'$ or $V^*$).

**Theorem** Let $\beta = \{v_\alpha\}_{\alpha \in \mathcal{A}}$ be a basis for $V$. Then

(i) any linear functional $\ell : V \to \mathbb{F}$ is uniquely determined by its restriction to $\beta$;

(ii) any function $f : \beta \to \mathbb{F}$ can be (uniquely) extended to a linear functional on $V$.

Thus we have a one-to-one correspondence between elements of the dual space $V'$ and collections of scalars $c_\alpha$, $\alpha \in \mathcal{A}$. Namely, $\ell \mapsto \{\ell(v_\alpha)\}_{\alpha \in \mathcal{A}}$. 
Dual basis

Let $\beta = [v_1, v_2, \ldots, v_n]$ be a basis for a vector space $V$. For any $1 \leq i \leq n$ let $f_i$ denote a unique linear functional on $V$ such that $f_i(v_j) = 1$ if $i = j$ and 0 otherwise.

If $v = r_1v_1 + r_2v_2 + \cdots + r_nv_n$, then $f_i(v) = r_i$. That is, the functional $f_i$ evaluates the $i$th coordinate of the vector $v$ relative to the basis $\beta$.

**Theorem** The functionals $f_1, f_2, \ldots, f_n$ form a basis for the dual space $V'$ (called the dual basis of $\beta$).
Double dual space

The **double dual** of a vector space $V$ is $V''$, the dual of $V'$. Since $V'$ is a functional vector space, to any vector $v \in V$ we associate an evaluation mapping, denoted $\hat{v}$, given by $\hat{v}(f) = f(v)$, $v \in V$. This mapping is linear, hence it is an element of $V''$.

**Theorem** Consider a mapping $\chi : V \rightarrow V''$ given by $\chi(v) = \hat{v}$. Then

(i) $\chi$ is linear;
(ii) $\chi$ is one-to-one;
(iii) $\chi$ is onto if and only if $\dim V < \infty$.

**Corollary 1** If $V$ is finite-dimensional, then $\chi$ is an isomorphism of $V$ onto $V''$.

**Corollary 2** If $V$ is finite-dimensional, then any basis for $V'$ is the dual basis of some basis for $V$. 
Dual linear transformation

Suppose \( V \) and \( W \) are vector spaces and \( L : V \to W \) is a linear transformation. The **dual transformation** of \( L \) is a transformation \( L' : W' \to V' \) given by \( L'(f) = f \circ L \). That is, \( L' \) precomposes each linear functional on \( W \) with \( L \).

It is easy to see that \( L'(f) \) is indeed a linear functional on \( V \). Also, \( L' \) is linear.

Suppose \( V \) and \( W \) are finite-dimensional. Let \( \beta \) be a basis for \( V \) and \( \gamma \) be a basis for \( W \). Let \( \beta' \) be the dual basis of \( \beta \) and \( \gamma' \) be the dual basis for \( \gamma \).

**Theorem** If \( [L]_{\beta}^{\gamma} = A \) then \( [L']_{\gamma'}^{\beta'} = A^t \).
Dual of an inner product space

Let $V$ be a vector space with an inner product $\langle \cdot, \cdot \rangle$. For any $y \in V$ consider a function $\ell_y : V \rightarrow F$ given by $\ell_y(x) = \langle x, y \rangle$ for all $x \in V$. This function is linear.

**Theorem**  Let $\theta : V \rightarrow V'$ be given by $\theta(v) = \ell_v$. Then (i) $\theta$ is linear if $F = \mathbb{R}$ and half-linear if $F = \mathbb{C}$; (ii) $\theta$ is one-to-one.

**Corollary**  If $V$ is finite-dimensional, then any linear functional on $V$ is uniquely represented as $\ell_v$ for some $v \in V$. 
Adjoint operator

Let $L$ be a linear operator on an inner product space $V$.

**Definition.** The adjoint of $L$ is a transformation $L^* : V \rightarrow V$ satisfying $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$ for all $x, y \in V$.

Notice that the adjoint of $L$ may not exist.

**Theorem 1** If the adjoint $L^*$ exists, then it is unique and linear.

**Theorem 2** If $V$ is finite-dimensional, then the adjoint operator $L^*$ always exists.

**Properties of adjoint operators:**

- $(L_1 + L_2)^* = L_1^* + L_2^*$
- $(rL)^* = \overline{r} L^*$
- $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$
- $(L^*)^* = L$
- $\text{id}_V^* = \text{id}_V$