Lecture 34:
Unitary operators.
Orthogonal matrices.
Diagonalization of normal operators

**Theorem** A linear operator $L$ on a finite-dimensional inner product space $V$ is normal if and only if there exists an orthonormal basis for $V$ consisting of eigenvectors of $L$.

**Corollary 1** Suppose $L$ is a normal operator. Then

(i) $L$ is self-adjoint if and only if all eigenvalues of $L$ are real ($\lambda = \overline{\lambda}$);

(ii) $L$ is anti-selfadjoint if and only if all eigenvalues of $L$ are purely imaginary ($\overline{\lambda} = -\lambda$);

(iii) $L$ is unitary if and only if all eigenvalues of $L$ are of absolute value 1 ($\overline{\lambda} = \lambda^{-1}$).

*Idea of the proof:* $L(x) = \lambda x \iff L^*(x) = \overline{\lambda} x$.

**Corollary 2** A linear operator $L$ on a finite-dimensional, real inner product space $V$ is self-adjoint if and only if there exists an orthonormal basis for $V$ consisting of eigenvectors of $L$. 

Diagonalization of normal matrices

**Theorem**  Matrix $A \in M_{n,n}(\mathbb{C})$ is normal if and only if there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of $A$.

**Corollary 1**  Suppose $A \in M_{n,n}(\mathbb{C})$ is a normal matrix. Then

(i) $A$ is Hermitian if and only if all eigenvalues of $A$ are real;
(ii) $A$ is skew-Hermitian if and only if all eigenvalues of $A$ are purely imaginary;
(iii) $A$ is unitary if and only if all eigenvalues of $A$ are of absolute value 1.

**Corollary 2**  Matrix $A \in M_{n,n}(\mathbb{R})$ is symmetric if and only if there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$. 
Example. $A_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi A_\psi = A_{\phi+\psi}$

$$
A_\phi A_\psi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} = 
\begin{pmatrix} 
\cos \phi \cos \psi - \sin \phi \sin \psi & -\cos \phi \sin \psi - \sin \phi \cos \psi \\
\sin \phi \cos \psi + \cos \phi \sin \psi & \cos \phi \cos \psi - \sin \phi \sin \psi
\end{pmatrix}
= \begin{pmatrix} 
\cos(\phi + \psi) & -\sin(\phi + \psi) \\
\sin(\phi + \psi) & \cos(\phi + \psi)
\end{pmatrix} = A_{\phi+\psi}.
$$

- $A_0 = I$

$$
A_0 = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.
$$
Example. \( A_{\phi} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \), \( \phi \in \mathbb{R} \).

- \( A_{\phi}^{-1} = A_{-\phi} \)

\[ A_{\phi}A_{-\phi} = A_{\phi+(-\phi)} = A_0 = I \implies A_{\phi}^{-1} = A_{-\phi}. \]

- \( A_{-\phi} = A_{\phi}^t \)

\[ A_{-\phi} = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = A_{\phi}^t. \]

- \( A_{\phi} \) is orthogonal

\[ A_{\phi}^t = A_{-\phi} = A_{\phi}^{-1} \implies A_{\phi} \) is orthogonal.
Example. \( A_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \), \( \phi \in \mathbb{R} \).

Characteristic polynomial:
\[
\det(A_\phi - \lambda) = \begin{vmatrix} \cos \phi - \lambda & -\sin \phi \\ \sin \phi & \cos \phi - \lambda \end{vmatrix} = (\cos \phi - \lambda)^2 + \sin^2 \phi.
\]

Eigenvalues: \( \lambda_1 = \cos \phi + i \sin \phi = e^{i\phi} \),
\( \lambda_2 = \cos \phi - i \sin \phi = e^{-i\phi} \).

Associated eigenvectors: \( \mathbf{v}_1 = (1, -i)^t \), \( \mathbf{v}_2 = (1, i)^t \).

\[
A_\phi \mathbf{v}_1 = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos \phi + i \sin \phi \\ \sin \phi - i \cos \phi \end{pmatrix} = \lambda_1 \mathbf{v}_1.
\]

Note that \( \lambda_2 = \overline{\lambda_1} \) and \( \mathbf{v}_2 = \overline{\mathbf{v}_1} \). Since the matrix \( A_\phi \) has real entries, \( A_\phi \mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \) implies \( A_\phi \overline{\mathbf{v}_1} = \overline{\lambda_1} \overline{\mathbf{v}_1} \).

We have \( \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 1 \cdot 1 + (-i) \cdot i = 1 + (-i)^2 = 0 \),
\( \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 2 \). Hence vectors \( \frac{1}{\sqrt{2}} \mathbf{v}_1 \) and \( \frac{1}{\sqrt{2}} \mathbf{v}_2 \) form an orthonormal basis for \( \mathbb{C}^2 \).
Characterization of unitary matrices

**Theorem** Given an $n \times n$ matrix $A$ with complex entries, the following conditions are equivalent:

(i) $A$ is unitary: $A^* = A^{-1}$;
(ii) columns of $A$ form an orthonormal basis for $\mathbb{C}^n$;
(iii) rows of $A$ form an orthonormal basis for $\mathbb{C}^n$.

*Sketch of the proof:* Entries of the matrix $A^*A$ are inner products of columns of $A$. Entries of $AA^*$ are inner products of rows of $A$. It follows that $A^*A = I$ if and only if the columns of $A$ form an orthonormal set. Similarly, $AA^* = I$ if and only if the rows of $A$ form an orthonormal set.

The theorem implies that a unitary matrix is the transition matrix changing coordinates from one orthonormal basis to another.
Diagonalization of normal matrices: revisited

**Theorem 1** Given an $n \times n$ matrix $A$ with complex entries, the following conditions are equivalent:

(i) $A$ is normal: $A^* A = AA^*$;

(ii) there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of $A$;

(iii) there exists a diagonal matrix $D$ and a unitary matrix $U$ such that $A = U D U^{-1}$ ($= U D U^*$).

**Theorem 2** Given an $n \times n$ matrix $A$ with real entries, the following conditions are equivalent:

(i) $A$ is symmetric: $A^t = A$;

(ii) there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A$;

(iii) there exists a diagonal matrix $D$ (with real entries) and an orthogonal matrix $U$ such that $A = U D U^{-1}$ ($= U D U^t$).
Characterizations of unitary operators

**Theorem** Given a linear operator on a finite-dimensional inner product space $V$, the following conditions are equivalent:

(i) $L$ is unitary;
(ii) $\langle L(x), L(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$;
(iii) $\|L(x)\| = \|x\|$ for all $x \in V$;
(iv) the matrix of $A$ relative to an orthonormal basis is unitary;
(v) $L$ maps some orthonormal basis for $V$ to another orthonormal basis;
(vi) $L$ maps any orthonormal basis for $V$ to another orthonormal basis.

*Proof that $(i) \implies (ii)$:* $\langle L(x), L(y) \rangle = \langle x, L^*(L(y)) \rangle = \langle x, y \rangle$. 