Problem. Let $R$ denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $R(e_1) = e_3$, $R(e_2) = e_1$, $R(e_3) = e_2$.

(i) Is $R$ a rotation about an axis? Is $R$ a reflection in a plane? Explain your answers.

(ii) If $R$ is a rotation, find the axis and the angle. If $R$ is a reflection, find the plane. If $R$ is neither rotation nor reflection, describe the action of $R$ in geometric terms.

The matrix of the operator $R$ (relative to the standard basis) is

$$
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

This matrix is orthogonal. Therefore $R$ is a rigid motion. According to the classification, $R$ is either a rotation about an axis, or a reflection in a plane, or the composition of two. Since $\det A = 1 > 0$, $R$ is a rotation.

As $R$ is a rotation about an axis, the matrix $A$ is similar to

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix},
$$

where $\phi$ is the angle of rotation. Similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). The trace of $A$ is 0. Hence $1 + 2 \cos \phi = 0$. Then $\cos \phi = -1/2$ so that $\phi = 2\pi/3$.

The axis of the rotation $R$ is the set of all points fixed by $R$. For any vector $(x, y, z) \in \mathbb{R}^3$ we have

$$
R(x, y, z) = R(xe_1 + ye_2 + ze_3) = xR(e_1) + yR(e_2) + zR(e_3) = xe_3 + ye_1 + ze_2 = (y, z, x).
$$

Therefore $R(x, y, z) = (x, y, z)$ if and only if $x = y = z$. Thus the axis of the rotation is the line spanned by the vector $(1, 1, 1)$. 

Quiz 3: Solution