Test 1: Solutions

Problem 1 (20 pts.) Determine which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set $S_1$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y - z = 0$.
(iii) The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x^2 - y^2 = 0$.
(iv) The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $e^x + e^y = 0$.

Solution: $S_2$ and $S_2'$ are subspaces of $\mathbb{R}^3$, the other sets are not.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

The set $S_1$ is the union of three planes $x = 0$, $y = 0$, and $z = 0$. It is not closed under addition as the following example shows: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

$S_2$ is a plane passing through the origin. It is easy to check that $S_2$ is closed under addition and scalar multiplication. Alternatively, $S_2$ is a subspace of $\mathbb{R}^3$ since it is the null-space of a linear functional $\ell : \mathbb{R}^3 \to \mathbb{R}$ given by $\ell(x, y, z) = x + y - z$, $(x, y, z) \in \mathbb{R}^3$.

$S_2'$ is a subspace of $\mathbb{R}^3$ since it is the null-space of a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$
L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y - z \\ 2y - 3z \end{pmatrix}
$$

for all $x, y, z \in \mathbb{R}$.

Since $x^2 - y^2 = (x - y)(x + y)$, the set $S_3$ is the union of two planes $x - y = 0$ and $x + y = 0$. The following example shows that $S_3$ is not closed under addition: $(1, 1, 0) + (1, -1, 0) = (2, 0, 0)$.

The set $S_4$ is the intersection of two planes $2y - 3z = 0$ and $2x - 3y = 1$. Hence $S_4$ is a line. One of the planes does not pass through the origin so that $S_4$ does not contain the zero vector. Therefore this set is not a subspace.

Since $e^x > 0$ for any $x \in \mathbb{R}$, the set $S_4'$ is empty. The empty set is not a subspace.

Thus $S_2$ and $S_2'$ are subspaces of $\mathbb{R}^3$ while $S_1$, $S_3$, $S_4$, and $S_4'$ are not.

Problem 2 (25 pts.) Let $W$ be a subspace of $\mathcal{M}_{2,2}(\mathbb{R})$ spanned by matrices $A, A^2, A^3, \ldots, A^n, \ldots$, where

$$
A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.
$$

Find a basis for $W$, then extend it to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$.

Solution: $\{A, A^2\}$ is a basis for $W$; the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ extend it to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$. 

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First we compute several powers of the matrix $A$:

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

Since $A^3 = I$, we have $A^{k+3} = A^k A^3 = A^k$ for any integer $k > 0$. It follows that $A^{3m} = I$, $A^{1+3m} = A$, and $A^{2+3m} = A^2$ for any integer $m > 0$. Therefore the subspace $W$ is spanned by the matrices $A$, $A^2$, and $A^3 = I$. Further, we have $A + A^2 + A^3 = 0$. Hence $A^3 = -A - A^2$, which implies that $A$ and $A^2$ span $W$ as well. Clearly, $A$ and $A^2$ are linearly independent. Therefore $\{A, A^2\}$ is a basis for $W$.

The matrices

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

form a basis for the vector space $\mathcal{M}_{2,2}(\mathbb{R})$. For example, we can extend the set $\{A, A^2\}$ to a basis for $\mathcal{M}_{2,2}(\mathbb{R})$ by adding two of these matrices. To verify this, it is enough to show that the matrices $A, A^2, E_1, E_2$ are linearly independent. Assume that $r_1 A + r_2 A^2 + r_3 E_1 + r_4 E_2 = 0$ for some scalars $r_1, r_2, r_3, r_4 \in \mathbb{R}$. Since

$$r_1 A + r_2 A^2 + r_3 E_1 + r_4 E_2 = r_1 \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} + r_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -r_1 + r_3 & r_1 - r_2 + r_4 \\ -r_1 + r_2 & -r_2 \end{pmatrix},$$

we have $-r_1 + r_3 = r_1 - r_2 + r_4 = -r_1 + r_2 = -r_2 = 0$. It follows that $r_1 = r_2 = r_3 = r_4 = 0$. Thus $A, A^2, E_1, E_2$ are linearly independent.

**Problem 3 (20 pts.)** Let $V_1$, $V_2$, and $V_3$ be finite-dimensional vector spaces. Suppose that $L : V_1 \rightarrow V_2$ and $T : V_2 \rightarrow V_3$ are linear transformations. Prove that $\text{rank}(T \circ L) \leq \text{rank}(L)$ and $\text{rank}(T \circ L) \leq \text{rank}(T)$.

Since $(T \circ L)(x) = T(L(x))$ for any $x \in V_1$, it follows that the range of the composition $T \circ L$ is contained in the range of $T$: $\mathcal{R}(T \circ L) \subseteq \mathcal{R}(T)$. Then $\dim \mathcal{R}(T \circ L) \leq \dim \mathcal{R}(T)$, that is, $\text{rank}(T \circ L) \leq \text{rank}(T)$.

By the Dimension Theorem, $\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim \mathcal{R}(T \circ L) + \dim \mathcal{N}(T \circ L) = \dim V_1$. Since $\text{rank}(L) = \dim \mathcal{R}(L)$ and $\text{rank}(T \circ L) = \dim \mathcal{R}(T \circ L)$, the inequality $\text{rank}(T \circ L) \leq \text{rank}(L)$ is equivalent to the inequality $\dim \mathcal{N}(T \circ L) \geq \dim \mathcal{N}(L)$. We are going to prove the latter.

Let $0_i$ denote the zero vector in the vector space $V_i$, $1 \leq i \leq 3$. If $L(x) = 0_2$ for some vector $x \in V_1$, then $(T \circ L)(x) = T(L(x)) = 0_3$, which equals $0_3$ since the transformation $T$ is linear. This means that the null-space of $L$ is contained in the null-space of $T \circ L$: $\mathcal{N}(L) \subseteq \mathcal{N}(T \circ L)$. Consequently, $\dim \mathcal{N}(L) \leq \dim \mathcal{N}(T \circ L)$.

**Problem 4 (25 pts.)** The functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$ span a 4-dimensional subspace $V$ of the vector space $\mathcal{F}(\mathbb{R})$. Consider a linear transformation $D : V \rightarrow \mathcal{F}(\mathbb{R})$ given by $D(f) = f'$ for all functions $f \in V$.

(i) Show that the range of $D$ is $V$ and the null-space of $D$ is trivial.

(ii) Find the matrix of $D$ (regarded as an operator on $V$) relative to the basis $f_1, f_2, f_3, f_4$. 

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Since it is given that the functions \( f_1, f_2, f_3, f_4 \) span a 4-dimensional subspace, they must be linearly independent and form a basis for the subspace. First we compute the images of these functions under the transformation \( D \):

\[
(D f_1)(x) = f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),
\]

\[
(D f_2)(x) = f_2'(x) = (x \cos x)' = -x \sin x + \cos x = -f_1(x) + f_4(x),
\]

\[
(D f_3)(x) = f_3'(x) = (\sin x)' = \cos x = f_4(x),
\]

\[
(D f_4)(x) = f_4'(x) = (\cos x)' = -\sin x = -f_3(x).
\]

Since all four images are in \( V \), it follows that the entire range of \( D \) is contained in \( V \). Also, we can write down the matrix of \( D \) (regarded as an operator on \( V \)) relative to the basis \( f_1, f_2, f_3, f_4 \):

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0
\end{pmatrix}.
\]

To prove that the range \( \mathcal{R}(D) \) coincides with \( V \), it is enough to show that each of the functions \( f_1, f_2, f_3, f_4 \) is in \( \mathcal{R}(D) \). Indeed,

\[
D(-f_2 + f_3) = -D(f_2) + D(f_3) = -(-f_1 + f_4) + f_4 = f_1,
\]

\[
D(f_1 + f_4) = D(f_1) + D(f_4) = (f_2 + f_3) + (-f_3) = f_2,
\]

\[
D(-f_4) = -D(f_4) = -(-f_3) = f_3,
\]

\[
D(f_3) = f_4.
\]

By the Dimension Theorem, \( \dim \mathcal{R}(D) + \dim \mathcal{N}(D) = \dim V \). Since the range of \( D \) is \( V \), it follows that \( \dim \mathcal{N}(D) = 0 \). Thus the null-space \( \mathcal{N}(D) \) is trivial.

**Problem 4' (25 pts.)** The functions \( f_1(x) = x \sin x, f_2(x) = x \cos x, f_3(x) = \sin x, \) and \( f_4(x) = \cos x \) span a 4-dimensional subspace \( V \) of the vector space \( \mathcal{F}(\mathbb{R}) \). Consider a linear transformation \( L : V \to \mathcal{F}(\mathbb{R}) \) given by \((Lf)(x) = f(x + 1), x \in \mathbb{R}\) for all functions \( f \in V \).

(i) Show that the range of \( L \) is \( V \) and the null-space of \( L \) is trivial.

(ii) Find the matrix of \( L \) (regarded as an operator on \( V \)) relative to the basis \( f_1, f_2, f_3, f_4 \).

**Solution:** the matrix of \( L \) is

\[
\begin{pmatrix}
\cos 1 & -\sin 1 & 0 & 0 \\
\sin 1 & \cos 1 & 0 & 0 \\
\cos 1 & -\sin 1 & \cos 1 & -\sin 1 \\
\sin 1 & \cos 1 & \sin 1 & \cos 1
\end{pmatrix}.
\]

Since it is given that the functions \( f_1, f_2, f_3, f_4 \) span a 4-dimensional subspace, they must be linearly independent and form a basis for the subspace. First we compute the images of these functions under
the transformation $L$:

$$
(Lf_1)(x) = f_1(x + 1) = (x + 1)\sin(x + 1) = (x + 1)(\sin x \cos 1 + \cos x \sin 1) = \\
(\cos 1)f_1(x) + (\sin 1)f_2(x) + (\cos 1)f_3(x) + (\sin 1)f_4(x),
$$

$$
(Lf_2)(x) = f_2(x + 1) = (x + 1)\cos(x + 1) = (x + 1)(\cos x \cos 1 - \sin x \sin 1) = \\
(- \sin 1)f_1(x) + (\cos 1)f_2(x) + (- \sin 1)f_3(x) + (\cos 1)f_4(x),
$$

$$
(Lf_3)(x) = f_3(x + 1) = \sin(x + 1) = \sin x \cos 1 + \cos x \sin 1 = \\
(\cos 1)f_3(x) + (\sin 1)f_4(x),
$$

$$
(Lf_4)(x) = f_4(x + 1) = \cos(x + 1) = \cos x \cos 1 - \sin x \sin 1 = \\
(- \sin 1)f_3(x) + (\cos 1)f_4(x).
$$

Since all four images are in $V$, it follows that the entire range of $L$ is contained in $V$. Also, we can write down the matrix of $L$ (regarded as an operator on $V$) relative to the basis $f_1, f_2, f_3, f_4$:

$$
\begin{pmatrix}
\cos 1 & -\sin 1 & 0 & 0 \\
\sin 1 & \cos 1 & 0 & 0 \\
\cos 1 & -\sin 1 & \cos 1 & -\sin 1 \\
\sin 1 & \cos 1 & \sin 1 & \cos 1
\end{pmatrix}.
$$

It follows from the definition of the operator $L$ that the function $Lf$ is identically zero only if $f$ is identically zero. Hence the null-space of $L$ is trivial.

By the Dimension Theorem, $\dim \mathcal{R}(L) + \dim \mathcal{N}(L) = \dim V$. Since the null-space of $L$ is trivial, we have $\dim \mathcal{N}(L) = 0$ so that $\dim \mathcal{R}(L) = \dim V$. Since the range $\mathcal{R}(L)$ is contained in $V$, it follows that $\mathcal{R}(L) = V$.

**Bonus Problem 5 (15 pts.)** The set $\mathbb{R}_+$ of positive real numbers is a (real) vector space with respect to unusual operations of addition and scalar multiplication given by $x \oplus y = xy$ and $r \odot x = x^r$ for all $x, y \in \mathbb{R}_+$ and $r \in \mathbb{R}$. Prove that this vector space is isomorphic to $\mathbb{R}$ (with usual linear operations).

An isomorphism is provided by the logarithmic function $f(x) = \log x$ (to any base). Indeed, $f$ is a one-to-one mapping of $\mathbb{R}_+$ onto $\mathbb{R}$. Since $\log(xy) = \log x + \log y$ for any $x, y > 0$, we have $f(x \oplus y) = f(x) + f(y)$. Since $\log x^r = r \log x$ for any $x > 0$ and $r \in \mathbb{R}$, we have $f(r \odot x) = rf(x)$. Thus $f$ is a linear mapping.

**Bonus Problem 5’ (15 pts.)** Prove that the real numbers $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are linearly independent over $\mathbb{Q}$.

Assume that $a\sqrt{2} + b\sqrt{3} + c\sqrt{6} = 0$ for some rational numbers $a$, $b$, and $c$. We have to prove that $a = b = c = 0$.

Indeed, the equality $a\sqrt{2} + b\sqrt{3} + c\sqrt{6} = 0$ is equivalent to $a\sqrt{2} + b\sqrt{3} = -c\sqrt{6}$. Squaring both sides of the latter, we obtain $(a\sqrt{2} + b\sqrt{3})^2 = (-c\sqrt{6})^2$. After simplification, $2ab\sqrt{6} + 2a^2 + 3b^2 = 6c^2$. Since the numbers $2ab, 2a^2 + 3b^2$, and $6c^2$ are rational while $\sqrt{6}$ is not, it follows that $2ab = 0$. Then $a = 0$ or $b = 0$. In the first case, we have $b\sqrt{3} + c\sqrt{6} = 0$, which implies that $b = 0$ as otherwise $1/\sqrt{2} = -c/b$, a rational number. In the second case, we have $a\sqrt{2} + c\sqrt{6} = 0$, which implies that $a = 0$ as otherwise $1/\sqrt{3} = -c/a$, a rational number. Thus $a = b = 0$ in any case. Then $c = 0$ as well.